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SYSTEMS OF TWO LINEAR INTEGRAL EQUATIONS WITH TWO PARAMETERS AND SYMMETRIZABLE KERNELS.

By Margaret Buchanan.

1. **Introduction.**—Since the time of Sturm and Liouville, numerous memoirs have been written concerning the linear self-adjoint differential equation with a single parameter and with boundary conditions. In the equation

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + (\lambda q - r)u = 0 \tag{a}$$

p and its derivative p', q and r are continuous functions of x independent of the parameter λ, $p > 0$, $q > 0$, and r may have either sign in a given interval. Various writers—Sturm,* Liouville,† Bôcher,‡ Hilbert,§ Stekloff,|| Kneser,¶ Mason,** and others—have proved theorems concerning the existence of characteristic numbers λ_i, and the expansion of more or less arbitrarily given functions into series whose terms are the characteristic functions u_i.

The corresponding theory for a linear integral equation with one parameter and a symmetric kernel,

$$\varphi(s) = \lambda \int_a^b K(s, t)\varphi(t)dt,$$

was developed by Hilbert†† and Schmidt.‡‡ By the use of Green's function,

* Sturm, *Journal de Mathématiques*, Vol. 1 (1836), pp. 106–186.

† *Liouville's Journal*, Vol. 2 (1837), p. 16 and p. 418.

‡ Bôcher, "Encyklopädie der mathematischen Wissenschaften," II A, 7a; *Annals of Mathematics*, Ser. 2, Vol. 13 (1911), p. 71; *Comptes Rendus*, Vol. 140 (1905), p. 928.

§ Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," pp. 39–59.

|| Stekloff, *Annales de la Faculté des sciences de Toulouse*, Ser. 2, Vol. 3, pp. 281–313; *Comptes Rendus*, Vol. CL (1 Semester, 1910), pp. 601–603; ibid., id., pp. 452–454; ibid., Vol. CLI (2 Semester, 1910), pp. 800–802.

¶ Kneser, *Mathematische Annalen*, Vol. 58 (1904), p. 81; ibid., Vol. 60, p. 402; ibid., Vol. 63, p. 477.

** Mason, *Mathematische Annalen*, Vol. 58 (1904), p. 532; *Comptes Rendus*, Vol. 140 (1905), p. 1086; *Transactions of the American Mathematical Society*, Vol. 7 (1906), pp. 337–360; ibid., Vol. 13, p. 516.

†† Hilbert, loc. cit., pp. 46, 49.

‡‡ Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen," *Mathematische Annalen*, Vol. 63 (1906), p. 433.

Hilbert showed an important connection between a self-adjoint differential system and an integral equation with a symmetric kernel.

For the polar integral equation with a single parameter, that is, an equation with a kernel of the form $a(s)K(s, t)$, in which $a(s)$ is a continuous function that changes sign a finite number of times in the interval considered and $K(s, t)$ is symmetric and positive definite, Hilbert* developed a theory analogous to that for the orthogonal, or symmetric case. If q in equation (a) is a continuous function which changes sign a finite number of times in the interval, if $r \geq 0$ and $p > 0$, the equation leads, not to an orthogonal, but to a polar integral equation.† Hilbert's restriction that q vanish only a finite number of times in the interval has been removed by later writers.‡

A theory similar to that for the orthogonal integral equation has been developed for the integral equation with certain unsymmetric kernels of more general type by A. J. Pell,§ Marty,|| and others.

Systems of two linear self-adjoint differential equations with two parameters have been considered by Hilbert,¶ who gives an existence theorem for the equations

$$(b)\qquad \begin{aligned} &\frac{d\left(p\dfrac{dy}{dx}\right)}{dx} + (\lambda a + \mu b)y = 0 \qquad (x_1 \leq x \leq x_2),\\ &\frac{d\left(\pi\dfrac{d\eta}{d\xi}\right)}{d\xi} - (\lambda\alpha + \mu\beta)\eta = 0 \qquad (\xi_1 \leq \xi \leq \xi_2), \end{aligned}$$

when $p > 0$, $a > 0$, $\pi > 0$, $\alpha > 0$; p, a, b denoting analytic functions of x and π, α, β analytic functions of ξ. He states that if the further condition $\alpha b - a\beta = 0$ is fulfilled only for a finite number of analytic curves, an arbitrary function of x and ξ that satisfies certain conditions of continuity and the boundary conditions may be developed into a series in terms of the products $y_h(x)\eta_h(s)$, where y_h and η_h indicate simultaneous solutions of the given equations, satisfying the boundary conditions. For the case when

* Hilbert, loc. cit., p. 195.

† Hilbert, loc. cit., p. 205.

‡ Fubini, *Annali di Matematica*, Ser. 3, Vol. 17 (1910), p. 111. Marty, *Comptes Rendus*, Vol. CL (1910), p. 515; ibid., id., pp. 603–606. Garbe, *Mathematische Annalen*, Vol. 76 (1915), pp. 517–547.

§ Pell, *Bulletin of the American Mathematical Society*, Vol. 16 (1910), pp. 513–515; *Transactions of the American Mathematical Society*, Vol. 12 (1911), pp. 165–180.

|| Marty, *Comptes Rendus*, Vol. 150 (1910), p. 515; ibid., id., p. 605; ibid., id., p. 1031; ibid., id., p. 1499.

¶ Hilbert, loc. cit., p. 263. Oscillation theorems for this case have been proved by Yoshikawa, Göttingen Nachrichten, 1910, pp. 586–594; and Richardson, *Transactions of the American Mathematical Society*, Vol. 13, pp. 22–34.

$\alpha b - a\beta = 0$ nowhere in the region, every continuous function of x and ξ, with continuous first and second derivatives, is developable in terms of $y_h(x)\eta_h(s)$.

More recently A. J. Pell† has proved existence and expansion theorems for systems of linear equations with two parameters, of the types

$$(c)\qquad \begin{aligned} u_i &= \lambda \sum_{j=1}^{\infty} k_{ij}u_j + \mu \sum_{j=1}^{\infty} l_{ij}u_j, \\ v_k &= \lambda \sum_{l=1}^{\infty} m_{kl}v_l - \mu \sum_{l=1}^{\infty} n_{kl}v_l; \end{aligned}$$

and

$$(d)\qquad \begin{aligned} u(x) &= \lambda \int_a^b K(x, y)u(y)dy + \mu \int_a^b L(x, y)u(y)dy, \\ v(s) &= \lambda \int_c^d M(s, t)v(t)dt - \mu \int_c^d N(s, t)v(t)dt; \end{aligned}$$

where the matrices and kernels are symmetric.

In the present paper the theory indicated in the last paragraph is extended to equations of type (*d*) with symmetrizable kernels subject to certain conditions which are stated on page 6. Hilbert's results for the differential equations (*b*) are obtained by considering a special case. In § 2 it is shown that there exist real values of λ and μ for which the equations (*d*) with symmetrizable kernels, and the adjoint equations, have continuous solutions u_i, v_i and u_i^*, v_i^*, respectively. Some properties of the solutions are developed in § 3, and the coefficients $\{f_i\}$ of the expansion

$$f(x, s) = \sum_{i=1}^{n} f_i u_i(x)v_i(s)$$

are shown to be of finite norm. In § 4 the expansion of an arbitrary function of two variables is considered. An expansion

$$h(x, s) = \sum_{\alpha} w_\alpha^*(x, s) \int\int h(y, t)u_\alpha(y)v_\alpha(t)dydt,$$

in which

$$w_\alpha^*(x, s) = u_\alpha^*(x)\int N(t, s)v_\alpha^*(t)dt + v_\alpha^*(s)\int L(y, x)u_\alpha^*(y)dy,$$

is obtained for a function $h(x, s)$ that can be expressed by

$$\int\int [K^*(x, y)f(y, t)N^*(t, s) + L^*(x, y)f(y, t)M^*(t, s)]dydt$$

where $f(y, t)$ is any continuous function of y and t, and K^*, L^*, M^*, N^* are symmetric kernels connected with K, L, M, and N, respectively.

† Pell, "Linear Equations with Two Parameters," *Transactions of the American Mathematical Society*, Vol. 23, No. 2 (1922), pp. 198–211.

That functions satisfying less stringent conditions may be expanded in terms of other characteristic functions connected with the integral equations (*d*) is shown in § 5 for the two important cases where the kernels are symmetrizable functions of well-known types:

$$\text{I.} \quad K(x, y) = \int \mathbf{K}(x, \xi) T(\xi, y) d\xi, \qquad L(x, y) = \int \mathbf{L}(x, \xi) T(\xi, y) d\xi,$$

$$M(s, t) = \int \mathbf{M}(s, \eta) \overline{T}(\eta, t) d\eta, \qquad N(s, t) = \int \mathbf{N}(s, \eta) \overline{T}(\eta, t) d\eta.$$

$$\text{II.} \quad K(x, y) = a(x) T(x, y), \qquad L(x, y) = b(x) T(x, y),$$

$$M(s, t) = \bar{a}(s) \overline{T}(s, t), \qquad N(s, t) = \bar{b}(s) \overline{T}(s, t).$$

Kernels of this type have already been mentioned in connection with the polar integral equation. Differential equations (*b*) lead to integral equations with kernels of the second type.

In obtaining the results as to the existence of solutions and the expansion of arbitrary functions, the method used is the reduction of equations (*d*) with symmetrizable kernels to equations (*c*) with symmetric matrices and the application of the theory previously developed for equations in infinitely many variables with two parameters.

2. **Existence of Solutions.**—Consider the system of linear integral equations

$$(1) \qquad \begin{aligned} u(x) &= \lambda \int_a^b K(x, y) u(y) dy + \mu \int_a^b L(x, y) u(y) dy, \\ v(s) &= \lambda \int_c^d M(s, t) v(t) dt - \mu \int_c^d N(s, t) v(t) dt, \end{aligned}$$

and the adjoint system

$$(2) \qquad \begin{aligned} u^*(x) &= \lambda \int_a^b u^*(y) K(y, x) dy + \mu \int_a^b u^*(y) L(y, x) dy, \\ v^*(s) &= \lambda \int_c^d v^*(t) M(t, s) dt - \mu \int_c^d v^*(t) N(t, s) dt. \end{aligned}$$

In (1) and (2) the kernels K and L are real continuous functions of the real variables x and y in $a \leq x \leq b$, $a \leq y \leq b$, M and N are real continuous functions of the real variables s and t in $c \leq s \leq d$, $c \leq t \leq d$; L and N are positive definite* or have only positive characteristic numbers and $KN + LM \not\equiv 0$. Further, there exist continuous and symmetric positive definite kernels T and $\overline{T}$, such that K and L are symmetrizable on the left by T, M and N by $\overline{T}$, that is, $\int_a^b T(x, \xi) K(\xi, y) d\xi$, $\int_a^b T(x, \xi) L(\xi, y) d\xi$,

* The kernel L is positive definite if $\int \int f(x) L(x, y) f(y) > 0$, for functions f, not null functions, integrable and with squares integrable in the sense of Lebesgue, except on a set of points of measure zero.

$\int_c^d \overline{T}(s, \eta)M(\eta, t)d\eta$, $\int_c^d \overline{T}(s, \eta)N(\eta, t)d\eta$ are symmetric functions. We shall use the following notation:

$$\int_a^b T(x, \xi)K(\xi, y)d\xi = \int TK = K^*, \qquad \int TL = L^*,$$

$$\int_c^d \overline{T}(s, \eta)M(\eta, t)d\eta = \int \overline{T}M = M^*, \qquad \int \overline{T}N = N^*.$$

In general,

$$\int AB = \int A(x, z)B(z, y)dz, \qquad \int fA = \int f(y)A(y, x)dy,$$

$$\int Af = \int A(x, y)f(y)dy.*$$

To show that there exist real values of λ and μ for which systems (1) and (2) have continuous solutions, system (1) is reduced to a system of linear equations in infinitely many unknowns, with symmetric matrices. In the process of reduction, use is made of the expansion†

$$\int fg = \sum_{i=1}^{\infty} \int f\psi_i \int g\varphi_i \tag{3}$$

where f and g are continuous functions, g is of the form $\int Th$, and $\{\varphi_i, \psi_i\}$ is a closed biorthogonal system of continuous functions such that $\psi_i = \int T\varphi_i$. Let $\{\overline{\varphi}_k, \overline{\psi}_k\}$ be a second closed biorthogonal system of continuous functions such that $\overline{\psi}_k = \int \overline{T}\overline{\varphi}_k$. For $\{\varphi_i, \psi_i\}$ and $\{\overline{\varphi}_k, \overline{\psi}_k\}$ it is convenient to use functions related to the characteristic functions χ_i and $\overline{\chi}_k$ of the symmetric positive definite kernels T and $\overline{T}$, respectively. Since the characteristic numbers of T and $\overline{T}$ are positive,‡ we indicate them by a_i^2 and b_k^2 and write

$$\chi_i(x) = a_i^2 \int T(x, y)\chi_i(y)dy, \qquad \overline{\chi}_k(s) = b_k^2 \int \overline{T}(s, t)\overline{\chi}_k(t)dt,$$

which become

$$\frac{\chi_i(x)}{a_i} = \int T(x, y)a_i\chi_i(y)dy, \qquad \frac{\overline{\chi}_k(s)}{b_k} = \int \overline{T}(s, t)b_k\overline{\chi}_k(t)dt. \tag{4}$$

Let

$$\varphi_i = a_i\chi_i, \qquad \psi_i = \frac{\chi_i}{a_i}, \qquad \overline{\varphi}_k = b_k\overline{\chi}_k, \qquad \overline{\psi}_k = \frac{\overline{\chi}_k}{b_k},$$

---

* As we shall consider only the square $a \leq x \leq b$, $a \leq y \leq b$, for the variables x and y, and the square $c \leq s \leq d$, $c \leq t \leq d$, for the variables s and t, the limits of integration and statements as to the range of the variables will be omitted. Wherever there is no ambiguity, we shall omit the variables.

† Pell, "Biorthogonal Systems of Functions," *Transactions of the American Mathematical Society*, Vol. XII, p. 147.

‡ Lalesco, "Introduction à la Théorie des Équations Intégrales," p. 71.

then since we may assume that the systems $\{\chi_i\}$, $\{\overline{\chi}_k\}$ are orthogonal and normalized, we have

$$\int \varphi_i\psi_j = \delta_{ij} \quad \text{and} \quad \int \overline{\varphi}_k\overline{\psi}_l = \delta_{kl}, \quad \text{where} \quad \delta_{\xi\eta} = \begin{cases} 1 & \xi = \eta, \\ 0 & \xi \neq \eta. \end{cases}$$

The equations (4) give the relations

$$\psi_i(x) = \int T(x, y)\varphi_i(y)dy, \qquad \overline{\psi}_k(s) = \int \overline{T}(s, t)\overline{\varphi}_k(t)dt.$$

We now proceed to reduce the system (1) to a system of linear equations in infinitely many unknowns. Multiplying the first equation of (1) by $\psi_i(x)$, integrating, and then expanding the second member by (3), we have

$$\int u(x)\psi_i(x)dx = \sum_{i=1}^{\infty} \int\int \varphi_i(\xi) \int T(\xi, x)[\lambda K(x, y) \\ + \mu L(x, y)]dx \cdot \varphi_j(y)d\xi dy \cdot \int u(y)\psi_j(y)dy$$

or

$$\int u\psi_i = \sum_{j=1}^{\infty} \int\int \varphi_i \int T(\lambda K + \mu L)\varphi_j \cdot \int u\psi_j, \tag{5}$$

and similarly from the second equation of (1) we have

$$\int v\overline{\psi}_k = \sum_{l=1}^{\infty} \int\int \overline{\varphi}_k \int \overline{T}(\lambda M - \mu N)\overline{\varphi}_l \cdot \int v\overline{\psi}_l. \tag{6}$$

Equations (5) and (6) may be written in the form

$$\begin{aligned} x_i &= \lambda \sum_{j=1}^{\infty} k_{ij}x_j + \mu \sum_{j=1}^{\infty} l_{ij}x_j, \\ y_k &= \lambda \sum_{l=1}^{\infty} m_{kl}y_l - \mu \sum_{l=1}^{\infty} n_{kl}y_l, \end{aligned} \tag{7}$$

where

$$\begin{aligned} x_i &= \int u\psi_i, & k_{ij} &= \int\int \varphi_i K^*\varphi_j, & l_{ij} &= \int\int \varphi_i L^*\varphi_j, \\ y_k &= \int v\overline{\psi}_k, & m_{kl} &= \int\int \overline{\varphi}_k M^*\overline{\varphi}_l, & n_{kl} &= \int\int \overline{\varphi}_k N^*\overline{\varphi}_l. \end{aligned}$$

The matrices K, L, M, N of (7) are symmetric, since the kernels K^*, L^*, M^*, N^* are symmetric, and the sum of the squares of the elements of each is convergent. For example, the convergency of $\sum_{i,j} k_{ij}^2$ is evident from the following

$$k_{ij}^2 = \int\int \varphi_i(x) \int T(x, \xi)K(\xi, y)d\xi\varphi_j(y)dxdy \\ \times \int\int \varphi_i(x) \int T(y, \xi)K(\xi, y)d\xi \cdot \varphi_j(y)dxdy$$

$$= a_i \int\int \chi_i(x)K(\xi, x) \frac{\chi_j(\xi)}{a_j}\, dx d\xi \cdot a_j \int\int \chi_j(x)K(\xi, x) \frac{\chi_i(\xi)}{a_i}\, dx d\xi.$$

But

$$\int\int \chi_i K \chi_j \cdot \int\int \chi_j K \chi_i \leq [\int\int \chi_i K \chi_j]^2 + [\int\int \chi_j K \chi_i]^2,$$

and since†

$$\sum_{ij} \left[\int\int \chi_i(\xi)K(\xi, x)\chi_j(x) d\xi dx \right]^2 \leq \int\int [K(\xi, x)]^2 d\xi dx,$$

the series $\sum_{ij} k_{ij}^2$ is convergent.

The two matrices l_{ij} and n_{kl} are of positive definite type, if the kernels L^* and N^* are themselves positive definite. In proving L^* and N^* positive definite, use is made of the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$ defined by the relations‡

$$(8) \qquad \begin{aligned} \varphi_i &= \alpha_i^2 \int L\varphi_i, & \psi_i &= \alpha_i^2 \int \psi_i L, & \psi_i &= \int T\varphi_i, \\ \overline{\varphi}_k &= \beta_k^2 \int N\overline{\varphi}_k, & \overline{\psi}_k &= \beta_k^2 \int \overline{\psi}_k N, & \overline{\psi}_k &= \int \overline{T}\overline{\varphi}_k. \end{aligned}$$

By the expansion (3)

$$(9) \qquad \begin{aligned} &\int\int f(x)L^*(x, y)f(y)dxdy \\ &\quad = \sum_i \int\int T(x, \xi)f(x)dx \cdot \varphi_i(\xi)d\xi \cdot \int\int L(\xi, y)f(y)dy \cdot \psi_i(\xi)d\xi \\ &\quad = \sum_i \frac{[\int \psi_i(x)f(x)dx]^2}{\alpha_i^2}. \end{aligned}$$

Similarly,

$$\int\int g(s)N^*(s, t)g(t)dsdt = \sum_k \frac{[\int \overline{\psi}_k(s)g(s)ds]^2}{\beta_k^2}.$$

Therefore the kernels L^* and N^*, and consequently, since matrices obtained from the kernels by another set of biorthogonal functions corresponding to T and $\overline{T}$ would differ from K, L, M, N only by orthogonal matrices, the matrices l_{ij} and n_{kl} are positive definite.

Inasmuch as the matrices K, L, M, and N of (7) are symmetric and such that the sum of the squares of the elements of each one is convergent, inasmuch as L and N are positive definite and $k_{ij}n_{kl} + l_{ij}m_{kl} \not\equiv 0$, there exist real values of λ and μ for which the system (7) has solutions x_i, y_k of

† Hilbert, loc. cit., p. 181.

‡ Pell, "Existence Theorems for Certain Unsymmetric Kernels," *Bulletin of the American Mathematical Society*, Vol. 16, p. 515. Marty, *Comptes Rendus*, February 28 and April 25, 1910.

finite norm,† and to any pair of parameter values λ, μ there corresponds only a finite number of linearly independent solutions x_i, y_k.

We now pass from the equations (7) in infinitely many variables to the integral equations (2). By means of x_i, y_k, solutions of finite norm of (7) corresponding to a definite set of parameter values denoted by λ_0, μ_0, we obtain a pair of continuous solutions of (2) corresponding to the same parameter values λ_0, μ_0. Consider, in particular, the first equation of (7)

$$x_i = \sum_j \int\int \varphi_i(\xi)[\lambda_0 K^*(\xi, z) + \mu_0 L^*(\xi, z)]\varphi_j(z)d\xi dz \cdot x_j;$$

multiply both members by $\int [\lambda_0 K(y, x) + \mu_0 L(y, x)]\psi_i(y)dy$, and sum with respect to i. The result is

$$\sum_{i=1} \int [\lambda_0 K(y, x) + \mu_0 L(y, x)]\psi_i(y)dy \cdot x_i$$
$$= \sum_{ij} \int [\lambda_0 K(y, x) + \mu_0 L(y, x)]\psi_i(y)dy$$
$$\times \int\int \varphi_i(\xi)[\lambda_0 K^*(\xi, z) + \mu_0 L^*(\xi, z)]\varphi_j(z)d\xi dz \cdot x_j,$$

which reduces to

$$\sum_{i=1} \int [\lambda_0 K(w, x) + \mu_0 L(w, x)]\psi_i(w)dw \cdot x_i$$
$$(10) \qquad = \int [\lambda_0 K(y, x) + \mu_0 L(y, x)]$$
$$\times \sum_j \int [\lambda_0 K(w, y) + \mu_0 L(w, y)]\psi_j(w)dwdy \cdot x_j.$$

From (10) it is evident that for the parameter values or characteristic numbers λ_0 and μ_0, the continuous function

$$u^*(x) = \sum_j \int [\lambda_0 K(w, x) + \mu_0 L(w, x)]\psi_j(w)dw \cdot x_j$$

is a solution of

$$(2)' \qquad u^*(x) = \lambda_0 \int u^*(y)K(y, x)dy + \mu_0 \int u^*(y)L(y, x)dy,$$

which is the first equation of the adjoint system (2). Therefore by the Fredholm theory,

$$(1)' \qquad u(x) = \lambda_0 \int K(x, y)u(y)dy + \mu_0 \int L(x, y)u(y)dy$$

† Pell, *Transactions of the American Mathematical Society*, Vol. 23, No. 2, p. 203.

has a continuous solution $u(x)$, and by the same theory, if n denotes the number of linearly independent solutions $u_1^*, u_2^*, \cdots, u_n^*$ of (2)′, the equation (1)′ has exactly n linearly independent solutions $u_1, u_2, \cdots, u_n$. In the same way it may be shown from the second equation of (7) that for λ_0, μ_0 the second equation of the adjoint system (2) has a continuous solution $v^*(s)$. It follows that

$$v(s) = \lambda_0 \int M(s, t)v(t)dt - \mu_0 \int N(s, t)v(t)dt$$

has a continuous solution $v(s)$, and that, corresponding to a pair of characteristic numbers λ_0, μ_0, the number of linearly independent solutions v_k is the same as the number of linearly independent solutions v_k^*.

That a simple relation exists between the solutions of the systems (1) and (2) appears from the following: the equations (1) when multiplied by $T(x, \xi)$ and $\overline{T}(s, \eta)$, respectively, and integrated, become

$$\int T(x, \xi)u(x)dx = \lambda \int\int K(x, \xi)T(x, y)u(y)dydx + \mu \int\int L(x, \xi)T(x, y)u(y)dydx,$$

$$\int T(s, \eta)v(s)ds = \lambda \int\int M(s, \eta)\overline{T}(s, t)v(t)dtds - \mu \int\int N(s, \eta)\overline{T}(s, t)v(t)dtds,$$

and these equations show that $\int Tu$ and $\int \overline{T}v$ are solutions of the adjoint system (2). On account of the (1, 1) correspondence already noted between the solutions u and u^*, and between v and v^*, we may say

$$u_i^* = \int Tu_i, \qquad v_k^* = \int \overline{T}v_k.$$

These results give the following theorem:

THEOREM 1: *If $K(x, y)$, $L(x, y)$, $M(s, t)$, $N(s, t)$ are real continuous functions, if L and N are positive definite or have only positive characteristic numbers, and $KN + LM \not\equiv 0$, and if there exist continuous symmetric positive definite kernels T and $\overline{T}$ such that $\int TK$, $\int TL$, $\int \overline{T}M$, $\int \overline{T}N$ are symmetric, there exist values λ and μ, necessarily real, for which the system of equations* (1) *has continuous solutions u, v, not identically zero, and for which the adjoint system* (2) *has continuous solutions u^*, v^*, not identically zero. The solutions of systems* (1) *and* (2) *are connected by the relations $u_i^* = \int Tu_i$, $v_k^* = \int \overline{T}v_k$.*

3. **Properties of Solutions.**—Let $u_i(x)$, $v_i(s)$ be solutions of system (1) corresponding to λ_i, μ_i and $u_k(x)$, $v_k(s)$ solutions corresponding to λ_k, μ_k. Then from the equations for u_i and u_k, by using as multipliers $u_k^*(x)$ and $u_i^*(x)$ and integrating, we obtain

$$\int\int u_i T u_k = \lambda_i \int\int\int u_k TKu_i + \mu_i \int\int\int u_k TLu_i,$$

$$\int\int u_k T u_i = \lambda_k \int\int\int u_i TKu_k + \mu_k \int\int\int u_i TLu_k,$$

and therefore, since the kernels $\int TK$ and $\int TL$ are symmetric,

$$(11) \qquad (\lambda_i - \lambda_k)\int\int\int u_k TKu_i + (\mu_i - \mu_k)\int\int\int u_k TLu_i = 0.$$

In the same way, from the equations in v_i and v_k, there results

$$(12) \qquad (\lambda_i - \lambda_k)\int\int\int v_k \overline{T}Mv_i - (\mu_i - \mu_k)\int\int\int v_k \overline{T}Nv_i = 0.$$

If $\lambda_i \neq \lambda_k$ and $\mu_i \neq \mu_k$, the determinant of the coefficients of (11) and (12) must vanish, that is,

$$(13) \qquad \begin{vmatrix} \int\int u_k^* Ku_i & \int\int u_k^* Lu_i \\ \int\int v_k^* Mv_i & -\int\int v_k^* Nv_i \end{vmatrix} = 0.$$

The equations

$$\int\int u_i Tu_k = \lambda_i \int\int u_k TKu_i + \mu_i \int\int u_k TLu_i,$$
$$\int\int v_i \overline{T}v_k = \lambda_i \int\int v_k \overline{T}Mv_i - \mu_i \int\int v_k \overline{T}Nv_i,$$

give, on the elimination of μ and the use of the relation (13),

$$\begin{vmatrix} \int\int u_k^* Lu_i & \int u_k^* u_i \\ -\int\int v_k^* Nv_i & \int v_k^* v_i \end{vmatrix} = 0.$$

Hence the matrix

$$\begin{pmatrix} \int\int u_k^* Ku_i & \int\int u_k^* Lu_i & \int u_k^* u_i \\ \int\int v_k^* Mv_i & -\int\int v_k^* Nv_i & \int v_k^* v_i \end{pmatrix}$$

is of rank < 2, if $i \neq k$.

To show that $\int u_i u_i^* \int\int v_i^* Nv_i + \int v_i v_i^* \int\int u_i^* Lu_i \neq 0$, we have only to write the expression in the form

$$\int\int u_i Tu_i \int\int v_i N^* v_i + \int\int v_i \overline{T}v_i \int\int u_i L^* u_i,$$

where each term > 0 because of the positive definite character of T, $\overline{T}$, L^*, and N^*. Since $\int u_i u_i^* \int\int v_i^* Nv_i + \int v_i v_i^* \int\int u_i^* Lu_i > 0$ we may assume that the solutions u_i and v_i have been multiplied by such constants as to make

$$\int u_i u_i^* \int\int v_i^* Nv_i + \int v_i v_i^* \int\int u_i^* Lu_i = 1.$$

We have, therefore, for (u_i, v_i), (u_k, v_k), corresponding to different characteristic numbers, the relation

$$(14) \qquad \begin{aligned} &\int\int u_i u_k^* \int\int v_k^* Nv_i + \int v_i v_k^* \int\int u_k^* Lu_i \\ &\qquad = \int\int u_i(x) v_i(s) w_k^*(x, s)\,dx\,ds = \delta_{ik} \end{aligned}$$

where

$$(15) \qquad w_\alpha^*(x, s) = u_k^*(x)\int N(t, s) v_k^*(t)\,dt + v_k^*(s)\int L(y, x) u_k^*(y)\,dy.$$

From every system of n linearly independent solutions of the equations (1) corresponding to a particular pair of characteristic numbers it is possible to build up by linear combinations n linearly independent functions that satisfy the relation (14). Let $u_1, u_2, \cdots, u_n$; $v_1, v_2, \cdots, v_n$ be solutions of (1) corresponding to λ_0, μ_0. We assume, as above, that $\int\int u_1 v_1 w_1^* = 1$, and then determine c_i and d_i so that

$$\begin{aligned} &\int\int (u_i - c_i u_1)(v_i - d_i v_1) w_1^* \\ (16) \qquad &= \int\int u_i v_i w_1^* - c_i \int\int u_1 v_i w_1^* + d_i(c_i - \int\int u_i v_1 w_1^*) = 0, \\ &\qquad\qquad i = 2, 3, \cdots, n. \end{aligned}$$

If $\int\int u_i v_i w_1^* = 0$, take $c_i = d_i = 0$, and the above equation is satisfied. If $\int\int u_i v_i w_1^* \neq 0$, choose $c_i \neq \int\int u_i v_1 w_1^*$ and from (16) determine d_i. Let $u_1 = \bar{u}_1, u_i - c_i u_1 = \bar{u}_i$; $v_1 = \bar{v}_1, v_i - d_i v_1 = v_i$; $i = 2, 3, \cdots, n$. Then the new system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$; $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n$ is such that $\int\int \bar{u}_i \bar{v}_i \bar{w}_1^* = \delta_{i1}$. Proceeding as before, we determine $\bar{c}_i$ and $\bar{d}_i$ to satisfy the equation $\int\int (\bar{u}_i - \bar{c}_i \bar{u}_2)(\bar{v}_i - \bar{d}_i \bar{v}_2) \bar{w}_2^* = 0$, $i = 3, 4, \cdots, n$. By repeating the process indicated above, we obtain finally a system $U_1, U_2, \cdots, U_n$; $V_1, V_2, \cdots, V_n$, where U_i and V_i are linear homogeneous functions with constant coefficients of $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$, respectively, with the constants so determined that $\int\int U_i V_i W_k^* = \delta_{ik}$.

If the series

$$(17) \qquad f(x, s) = \sum_{i=1}^{\infty} f_i u_i(x) v_i(s)$$

is uniformly convergent, the coefficients of $u_i v_i$ may be expressed in a form analogous to the Fourier coefficients in the expansion

$$g(s) = \sum_{\nu} \varphi_\nu(s) \int_a^b g(t) \psi_\nu(t) dt,$$

where $\{\varphi_\nu, \psi_\nu\}$ is a biorthogonal system. Multiply (17) by $u_k^*(x) \int N(t, s) v_k^*(t) dt + v_k^*(s) \int L(y, x) u_k^*(y) dy$, integrate, and use the relation (14), thus obtaining

$$\begin{aligned} f_k &= \int\int\int u_k^*(x) f(x, s) N(t, s) v_k^*(t) dx ds dt + v_k^*(s) f(x, s) L(y, x) u_k^*(y) ds dx dy \\ &= \int\int f(x, s) w_k^*(x, s) dx ds. \end{aligned}$$

If the series

$$f(x, s) = \sum_{i=1}^{\infty} A_i w_i^*(x, s)$$

is uniformly convergent, the coefficients A_i may be expressed in terms of the solutions u_i and v_i. Multiplying the equation by $u_k(x) v_k(s)$ and

integrating, we have

$$\int\int f(x, s)u_k(x)v_k(s)dxds$$

$$= \sum_{i=1}^{\infty} A_i \left[\int v_k v_i^* \int\int u_i^* L u_k + \int u_k u_i^* \int\int v_i^* N v_k\right].$$

Hence

$$A_i = \int\int f(x, s)u_i(x)v_i(s)dxds.$$

Again, if a function $f(x, s)$ be expressed by a uniformly convergent series of the form

$$f(x, s) = \sum_i B_i u_i^*(x)v_i^*(s),$$

or

$$f(x, s) = \sum_i C_i w_i(x, s),$$

where

(18) $$w_i(x, s) = u_i(x)\int N(s, t)v_i(t)dt + v_i(s)\int L(x, y)v_i(y)dy,$$

the coefficients are found to be

$$B_i = \int\int f(x, s)w_i(x, s)dxds,$$

$$C_i = \int\int f(x, s)u_i^*(x)v_i^*(s)dxds.$$

Let $f(x, s)$ and $g(x, s)$ be continuous functions and suppose that the series on the right of (17) is uniformly convergent. After multiplication of (17) by $T(x, z)\overline{T}(s, \eta)[\int g(z, \xi)N(\eta, \xi)d\xi + \int g(\xi, \eta)L(z, \xi)d\xi]$, and integration, we have

$$\int\int\int\int f(x, s)T(x, z)N^*(s, \xi)g(z, \xi)dxdsd\xi dz$$

$$+ \int\int\int\int f(x, s)\overline{T}(s, \eta)L^*(x, \xi)g(\xi, \eta)dxdsd\eta d\xi$$

$$= \sum_i f_i\left[\int\int\int u_i^*(z)g(z, \xi)N(\eta, \xi)v_i^*(\eta)dzd\xi d\eta\right.$$

$$\left. + \int\int\int v_i^*(\eta)g(\xi, \eta)L(z, \xi)u_i^*(z)d\eta d\xi dz\right]$$

$$= \sum_i f_i g_i.$$

The coefficients $f_i = \int f w_i^*$ of a continuous function $f(x, s)$ are of finite norm. In order to prove that $\sum_i f_i^2$ is convergent, we establish, for $F(x, s)$

a continuous function, the relation

$$\int\int\int\int[F(x, s)L^*(x, y)F(y, t)\overline{T}(s, t) + F(x, s)N^*(s, t)F(y, t)T(x, y)]dxdsdtdy \geq 0. \tag{19}$$

This inequality is a direct result of the positive definite character of L^* and N^*, and is evident if the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$ defined by (8) are used in the expansion of the expression on the left of (19). By the expansion (3) and by substitutions from (8),

$$\begin{aligned}\int\int\int\int&[F(x, s)L^*(x, y)F(y, t)\overline{T}(s, t) \\ &\qquad + F(x, s)N^*(s, t)F(y, t)T(x, y)]dxdsdtdy \\ &= \sum_k \int\int\int F(x, s)\overline{\psi}_k(s)ds\cdot L^*(x, y)\int\int \overline{T}(s, t)F(y, t)dt\cdot\overline{\varphi}_k(s)dsdydx \\ &\quad + \sum_i \int\int\int F(x, s)\psi_i(x)dxN^*(s, t)\int\int T(x, y)F(y, t)dy\cdot\varphi_i(x)dxdtds \\ &= \sum_k \int\int\int F(x, s)\overline{\psi}_k(s)dsL^*(x, y)\int F(y, t)\overline{\psi}_k(t)dtdydx \\ &\quad + \sum \int\int\int F(x, s)\psi_i(x)dx\cdot N^*(s, t)\int F(y, t)\psi_i(y)dydtds,\end{aligned}$$

where each term is positive because L^* and N^* are positive definite Hence the relation (19) is established. To show $\sum_i f_i^2$ convergent, substitute $f(x, s) - \sum_i f_i u_i(x)v_i(s)$ for $F(x, s)$ in (19), thus obtaining

$$\begin{aligned}\int\int\int\int&[f(x, s)L^*(x, y)f(y, t)\overline{T}(s, t)+f(x, s)N^*(s, t)f(y, t)T(x, y)]dxdydsdt \\ &- \sum_i f_i\int\int\int\int[v_i^*(s)f(x, s)L(\xi, x)u_i^*(\xi)dsdxd\xi \\ &\qquad\qquad + u_i^*(x)f(x, s)N(\eta, s)v_i^*(\eta)dxdsd\eta] \\ &- \sum_i f_i\int\int\int\int[v_i^*(t)f(y, t)L(\xi, y)u_i^*(\xi)dtdyd\xi \\ &\qquad\qquad + u_i^*(y)f(y, t)N(\eta, t)v_i^*(\eta)dydtd\eta] \\ &+ \sum_i f_i^2\Big[\int u_i(x)u_i^*(x)dx\int\int v_i^*(\eta)N(\eta, s)v_i(s)d\eta ds \\ &\qquad\qquad + \int v_i(s)v_i^*(s)ds\int\int u_i^*(\xi)L(\xi, x)u_i(x)d\xi dx\Big] \geq 0.\end{aligned}$$

This reduces to

$$\int\int\int\int [fL^*f\overline{T} + fN^*fT] - 2\sum_i f_i^2 + \sum_i f_i^2 \geq 0,$$

which gives

$$\sum_i f_i^2 \leq \int\int\int\int [fL^*f\overline{T} + fN^*fT]. \tag{20}$$

We conclude from (20) that the sequence of coefficients $\{f_i\}$ is of finite norm.

4. **Expansion of Arbitrary Functions of Two Variables.**—For a system of two linear integral equations with symmetric kernels, it has been shown† that when f and g are continuous functions

$$\int\int\int\int [K(x, y)f(x, s)N(s, t)g(y, t)$$

$$+ L(x, y)f(x, s)M(s, t)g(y, t)]dxdydsdt = \sum_\alpha \frac{f_\alpha g_\alpha}{\lambda_\alpha},$$

where $f_\alpha = \int\int f(x, s)w_\alpha(x, s)dxds$, and that

$$\int\int K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)dydt = \sum_\alpha \frac{w_\alpha(x, s)f_\alpha}{\lambda_\alpha}.$$

Corresponding forms may be obtained for the symmetrizable case as a result of the expansion† of the determinant matrix

$$k_{ij}n_{kl} + l_{ij}m_{kl} = \sum_\alpha \frac{w_{\alpha ik}w_{\alpha jl}}{\lambda_\alpha}, \tag{21}$$

where $w_{\alpha ik} = y_{\alpha k}\sum_j l_{ij}x_{\alpha j} + x_{\alpha i}\sum_l n_{kl}y_{\alpha l}$, $x_{\alpha i}$ and $y_{\alpha k}$ denoting solutions of (7) corresponding to the characteristic numbers λ_α, μ_α. Since

$$w_{\alpha ik} = \int v_\alpha\overline{\psi}_k \sum_j \int\int \varphi_i L^*\varphi_j \int u_\alpha\psi_j + \int u_\alpha\psi_i \sum_l \int\int \varphi_k N^*\varphi_l \int v_\alpha\overline{\psi}_l,$$

by means of (21), the following equality may be verified:

$$\int\int\int\int K^*(x, y)f(x, s)N^*(s, t)g(y, t) \tag{22}$$

$$+ L^*(x, y)f(x, s)M^*(s, t)g(y, t)]dxdsdydt = \sum_\alpha \frac{f_\alpha g_\alpha}{\lambda_\alpha}.$$

As $f(x, s)$ and $g(x, s)$ are any two continuous functions, it follows from (22) that

$$K^*(x, y)N^*(s, t) + L^*(x, y)M^*(s, t) = \sum_\alpha \frac{w_\alpha^*(x, s)w_\alpha^*(y, t)}{\lambda_\alpha},$$

if the series on the right is uniformly convergent.

† Pell, *Transactions of the American Mathematical Society*, Vol. 23, No. 2, p. 208.

The problem of the expansion of arbitrary functions in terms of the solutions u_α and v_α, u^*_α and v^*_α, and in terms of w_α and w^*_α will now be considered. On account of (21), we have the relation

$$\int\int[K^*(x, y)f(y, t)N^*(s, t) \tag{23}$$
$$+ L^*(x, y)f(y, t)M^*(s, t)]dydt = \sum_\alpha \frac{w^*_\alpha(x, s)f_\alpha}{\lambda_\alpha},$$

if the series on the right is uniformly convergent. As $\{f_\alpha\}$ is known to be of finite norm (§ 3), the series $\sum_\alpha w^*_\alpha(x, s)f_\alpha/\lambda_\alpha$ is absolutely uniformly convergent if $\{w^*_\alpha(x, s)/\lambda_\alpha\}$ is of finite norm. In proving the convergence of

$$\sum_\alpha\left(\frac{w^*_\alpha(x, s)}{\lambda_\alpha}\right)^2$$

we employ again the two biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$, corresponding to the unsymmetric kernels L and N, respectively. If there exists a continuous function $A(x, y, s, t)$, such that

$$\int\int\int\int \varphi_i(x)\psi_j(y)A(x, y, s, t)\overline{\varphi}_k(s)\overline{\psi}_l(t)dxdydsdt = a_{ikjl}\frac{\alpha_j^2\beta_l^2}{\alpha_j^2+\beta_l^2}, \tag{24}$$

where

$$a_{ikjl} = k^*_{ij}\cdot\frac{\delta_{kl}}{\beta_k^2} + m^*_{kl}\cdot\frac{\delta_{ij}}{\alpha_i^2}$$
$$= \int\int\psi_i(y)K(y, x)\varphi_j(x)dydx\frac{\delta_{kl}}{\beta_k^2} + \int\int\overline{\psi}_k(t)M(t, s)\overline{\varphi}_l(s)dtds\cdot\frac{\delta_{ij}}{\alpha_i^2},$$

it follows immediately that $\{w^*_\alpha(x, s)/\lambda_\alpha\}$ is of finite norm, for

$$\frac{w^*_\alpha(x, s)}{\lambda_\alpha} = \int\int A(x, y, s, t)w^*_\alpha(y, t)dydt, \tag{25}$$

and since $\int\int fw^*_i = f_i$, $\{\int\int A(x, y, s, t)w^*_\alpha(y, t)dydt\}$ is of finite norm. The relation expressed by (25) may be verified by substituting in (25) the value of w^*_α given by (15), expanding the second member in terms of the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\varphi_k, \overline{\psi}_k\}$, and finally multiplying the equation by $\varphi_i(x)\overline{\varphi}_k(s)$ and integrating with respect to x and s. The result of these operations is an equality already established †

$$c^2_{ik}x^*_{ik} = \lambda_\alpha\sum_{jl}a_{ikjl}x^*_{jl} \tag{26}$$

in which $x^*_{ik} = \int u^*_\alpha\varphi_i\int v^*_\alpha\varphi_k$, $c^2_{ik} = \frac{1}{\alpha_i^2}+\frac{1}{\beta_k^2}$, and a_{ikjl} is the matrix already defined.

† Pell, *Transactions of the American Mathematical Society*, Vol. 23, No. 2, p. 200.

As the existence of $A(x, y, s, t)$ satisfying (24) is not assured, we introduce a transformation $A[f(x, s)]$, where f is of the form $\int\int Tg\overline{T}$, such that if

$$f_1(x, s) = A[f(x, s)]$$

then

$$\int\int f_1(x, s)\varphi_i(x)\overline{\varphi}_k(s)dxds = \sum_{jl}\frac{a_{ikjl}}{c_{jl}^2}\int\int f(y, t)\varphi_j(y)\overline{\varphi}_l(t)dydt. \tag{27}$$

The relation between the transformed function $A[f(x, s)]$ and $A(x, y, s, t)$ of (24), when the function exists, is given by

$$A[f(x, s)] = \int\int A(x, y, s, t)f(y, t)dydt, \tag{28}$$

which follows directly from (27).

The transformed function of $g^*(s)\int f^*(y)L(y, x)dy$ exists and is a continuous function, for it may be expressed in the following form, in which the series on the right are uniformly convergent.

$$\begin{aligned} &A[g^*(s)\int f^*(y)L(y, x)dy] \\ &= \sum_{ik}\frac{\int\psi_i(y)K(y, x)dy\int f(\xi)\psi_i(\xi)d\xi}{\alpha_i^2}\cdot\frac{\alpha_i^2}{\alpha_i^2+\beta_k^2}\overline{\psi}_k(s)\int g(\eta)\overline{\psi}_k(\eta)d\eta \\ &+\sum_{ik}\frac{\psi_i(x)\int f(\xi)\psi_i(\xi)d\xi}{\alpha_i^2}\cdot\frac{\beta_k^2}{\alpha_i^2+\beta_k^2}\int g(\eta)\overline{\psi}_k(\eta)d\eta\int\overline{\psi}_k(t)M(t, s)dt. \end{aligned} \tag{29}$$

The second member of this equation may be written as the product of factors less than unity or of the form $\int F\psi_i$ and $\int F_1\overline{\psi}_k$, F and F_1 denoting continuous functions. As a consequence of (3)

$$\sum_i\left[\int F\psi_i\right]^2 = \int\int FTF$$

and

$$\sum_k\left[\int F\overline{\psi}_k\right]^2 = \int\int F_1\overline{T}F_1.$$

Therefore the two series on the right of (29) converge uniformly. To verify (29) multiply it by $\varphi_j(x)\overline{\varphi}_l(s)$, integrate with respect to x and s, and then factor the second member, thus reducing (29) to (27) with $f = g^*\int f^*L$. The transformed function of $f^*\int g^*N$ exists also and is continuous. Therefore $A[w_\alpha^*(x, s)]$, which we shall denote by A_α, exists and is a continuous function. As

$$A[w_\alpha^*(x, s)] = \frac{w_\alpha^*(x, s)}{\lambda_\alpha}$$

because of (25), we prove that $\{w_\alpha^*(x, s)/\lambda_\alpha\}$ is of finite norm by proving $\sum_{\alpha=1}^{\infty} A_\alpha^2$ convergent.

By the series of operations that follow, we derive the inequality

$$\sum_{\alpha=1}^{n} A_\alpha^2 \leq \Phi^2 + \Psi^2,$$

a continuous function, thus showing the convergency of $\sum_{\alpha=1}^{\infty} A_\alpha^2$. From (29) we obtain

$$A\left[g^* \int f^* L\right] = \sum_{ik} \int f\psi_i \int g\bar{\psi}_k \left[\frac{\int \psi_i K}{\alpha_i^2} \cdot \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} \bar{\psi}_k + \frac{\psi_i \int \bar{\psi}_k M}{\alpha_i^2} \cdot \frac{\beta_k^2}{\alpha_i^2 + \beta_k^2}\right]. \tag{31}$$

Since the second member of (31) is in the form $a(b + c)$ and since $2a(b + c) < a^2 + 2(b^2 + c^2)$, it follows that

$$2\left|A\left[g^*(s) \int f^*(y) L(y, x) dy\right]\right| \leq \sum_{ik} \frac{[\int f(y)\psi_i(y)dy]^2 [\int g(\eta)\bar{\psi}_k(\eta)d\eta]^2}{\alpha_i^2} + \Phi^2(x, s), \tag{32}$$

where

$$\Phi^2(x, s) = 2 \sum_{ik}\left[\left(\int \psi_i(y) K(y, x) dy\right)^2 \cdot \frac{\bar{\psi}_k^2(s)}{\alpha_i^2 + \beta_k^2} + \frac{\psi_i^2(x)}{\alpha_i^2}\left(\int \bar{\psi}_k(t) M(t, s) dt\right)^2\right].$$

The uniform convergence of $\sum_k \frac{\bar{\psi}_k^2}{\beta_k^2}$ and $\sum_i \frac{\psi_i^2}{\alpha_i^2}$ is given by Mercer's theorem,†

$$\sum_i \left[\int \psi_i K\right]^2 = \int KK^* \quad \text{and} \quad \sum_k \left[\int \bar{\psi}_k M\right]^2 = \int MM^*$$

by (3), hence Φ^2 is a continuous function. Because of (9) and (3), the inequality (32) becomes

$$2|A[g^*(s) \int f^*(y) L(y, x) dy]| \leq \int \int f(x) L^*(x, y) f(y) dx dy \cdot \int g(s) g^*(s) ds + \Phi^2(x, s). \tag{33}$$

† Mercer, "Symmetrizable Functions and their Expansion in Terms of Biorthogonal Functions," *Proceedings of the Royal Society of London*, Series A, Vol. XCVII, p. 409.

It may likewise be shown that

$$
(34)\quad \begin{aligned} &2|A[f^*(x)\textstyle\int g^*(t)N(t, s)dt]| \\ &\qquad \leq \textstyle\int\int g(s)N^*(s, t)g(t)dsdt\cdot\int f(x)f^*(x)dx + \Psi^2(x, s), \end{aligned}
$$

where Ψ^2 is a continuous function of x and s. From (33) and (34) there results

$$
(35)\quad \begin{aligned} &\Phi^2(x, s) - 2A[\textstyle\int \overline{T}(\eta, s)\int L(y, x)\int T(y, \xi)f(\xi, \eta)d\xi dy d\eta] \\ &\quad + \textstyle\int\int f(x, s)L^*(x, y)f(y, \eta)dxdy\cdot\int\int \overline{T}(\eta, s)d\eta ds \\ &\quad - 2A[\textstyle\int T(\xi, x)\int N(t, s)\int \overline{T}(\eta, t)f(\xi, \eta)d\xi d\eta dt] \\ &\quad + \textstyle\int\int f(x, s)N^*(s, t)f(\xi, t)dsdt\int\int T(\xi, x)d\xi dx + \Psi^2(x, s) \geq 0. \end{aligned}
$$

For $f(x, s)$ in (35) substitute $\sum_{\alpha=1}^{n} A_\alpha u_\alpha(x)v_\alpha(s)$, where $A_\alpha = \dfrac{w_\alpha^*(y, t)}{\lambda_\alpha}$. Then

$$
\begin{aligned} &2A\left[\int \overline{T}\int L\int Tf\right] + 2A\left[\int T\int N\int \overline{T}f\right] \\ &\qquad = 2A\left[\sum_\alpha v_\alpha^*\int Lu_\alpha^* + u_\alpha^*\int Nv_\alpha^*\right]\cdot A_\alpha = 2A_\alpha^2, \end{aligned}
$$

and

$$
\begin{aligned} &\int\int fL^*f\int\int \overline{T} + \int\int fN^*f\int\int T \\ &\qquad = \sum_\alpha A_\alpha^2\left(\int v_\alpha v_\alpha^*\int\int u_\alpha^*Lu_\alpha + \int u_\alpha u_\alpha^*\int\int v_\alpha^*Nv_\alpha\right) = \sum_\alpha A_\alpha^2. \end{aligned}
$$

Therefore (35) reduces to

$$
\Phi^2(x, s) - 2\sum_\alpha A_\alpha^2 + \sum_\alpha A_\alpha^2 + \psi^2(x, s) \geq 0,
$$

that is,

$$
\sum_{\alpha=1}^{n} A_\alpha^2 \leq \Phi^2 + \Psi^2.
$$

On account of the continuity of $\Phi^2 + \Psi^2$,

$$
\sum_{\alpha=1}^{n} A_\alpha^2 \leq P,
$$

where P denotes a constant, and $\{w_\alpha^*(x, s)/\lambda_\alpha\}$ is of finite norm for all values of x and s in the given intervals.

As f_α is known to be of finite norm (§ 3), we conclude that the series formed from the absolute values of the terms of

$$\sum_\alpha \frac{w_\alpha^*(x, s)f_\alpha}{\lambda_\alpha},$$

where $f(x, s)$ is any continuous function, is uniformly convergent. By (23)

$$h(x, s) = \int\int [K^*(x, y)f(y, t)N^*(t, s) + L^*(x, y)f(y, t)M^*(t, s)]dydt$$
$$= \sum_\alpha \frac{w_\alpha^*(x, s)f_\alpha}{\lambda_\alpha},$$

and the equality

$$\int\int h(y, t)u_\alpha(y)v_\alpha(t)dydt = \frac{f_\alpha}{\lambda_\alpha}$$

follows directly from (22), hence

$$h(x, s) = \sum_\alpha \frac{w_\alpha^*(x, s)f_\alpha}{\lambda_\alpha} = \sum_\alpha w_\alpha^*(x, s)\int\int h(y, t)u_\alpha(y)v_\alpha(t)dydt. \tag{36}$$

These results may be expressed as a theorem:

THEOREM 2: *If K, L, M, and N are continuous real functions such that K^*, L^*, M^*, and N^* are symmetric, if L and N are positive definite, and if $KN + LM \not\equiv 0$, any function $h(x, s)$ that can be expressed in the form*

$$h(x, s) = \int\int[K^*(x, y)f(y, t)N^*(t, s) + L^*(x, y)f(y, t)M^*(t, s)]dydt,$$

where $f(x, s)$ is any continuous function of x and s, may be expanded into the absolutely uniformly convergent series

$$h(x, s) = \sum_\alpha w_\alpha^*(x, s)\int\int h(y, t)u_\alpha(y)v_\alpha(t)dydt.$$

Equation (36) may be written in the form

$$\int\int\int\int T(x, \xi)[K(\xi, y)f(y, t)N(\eta, t) + L(\xi, y)f(y, t)M(\eta, t)]\overline{T}(\eta, t)d\xi d\eta dydt$$
$$= \sum_\alpha \int\int T(x, \xi)w_\alpha(\xi, \eta)\overline{T}(\eta, t)d\xi d\eta \int\int\int\int [K^*(x, y)f(y, t)N^*(s, t)$$
$$+ L^*(x, y)f(y, t)M^*(s, t)]u_\alpha^*(x)v_\alpha^*(s)dxdsdydt.$$

Inasmuch as T and $\overline{T}$ are definite, it follows that

$$\int\int [K(\xi, y)f(y, t)N(\eta, t) + L(\xi, y)f(y, t)M(\eta, t)]dydt$$
$$= \sum_\alpha w_\alpha(\xi, \eta) \int\int\int\int [K(\xi_1, y)f(y, t)N(\eta_1, t) + L(\xi_1, y)f(y, t)M(\eta_1, t)]dydt \cdot u_\alpha^*(\xi_1)v_\alpha^*(\eta_1)d\xi_1 d\eta_1$$

if the series is uniformly convergent. That is, if

$$g(x, s) = \int\int [K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)]dydt, \tag{37}$$

then

$$g(x, s) = \sum_\alpha w_\alpha(x, s) \int\int g(\xi, \eta)u_\alpha^*(\xi)v_\alpha^*(\eta)d\xi d\eta \tag{38}$$

if the series converges uniformly.

5. **Special Kernels.**—We shall now consider as special cases the equations (1) and (2) when K, L, M, and N are symmetrizable kernels of two important types.

Case I: $K(x, y) = \int \boldsymbol{K}(x, \xi)T(\xi, y)d\xi$, $\quad L(x, y) = \int \boldsymbol{L}(x, \xi)T(\xi, y)d\xi$,

$M(s, t) = \int \boldsymbol{M}(s, \eta)\overline{T}(\eta, t)d\eta$, $\quad N(s, t) = \int \boldsymbol{N}(s, \eta)\overline{T}(\eta, t)d\eta$,

where $\boldsymbol{K}$ and $\boldsymbol{M}$ are symmetric, and $\boldsymbol{L}$, $\boldsymbol{N}$, T, and $\overline{T}$ are symmetric and positive definite.

Case II: $K(x, y) = a(x)T(x, y)$, $\quad L(x, y) = b(x)T(x, y)$,

$M(s, t) = \bar{a}(s)\overline{T}(s, t)$, $\quad N(s, t) = \bar{b}(s)\overline{T}(s, t)$,

where $a(x)$, $b(x)$, $\bar{a}(s)$, $\bar{b}(s)$ are continuous functions and $b(x) > 0$, $\bar{b}(s) > 0$ for x and s in their respective intervals. The functions $T(x, y)$ and $\overline{T}(s, t)$ are symmetric and positive definite. The kernels L and N have only positive characteristic numbers.

Expansion Theorem for Case I.—For this case it will be shown that the series in (38) is uniformly convergent. Substituting for g in the second member of (38) the expression given by (37), we obtain

$$g(x, s) = \sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha} \int\int f(y, t)w_\alpha^*(y, t)dydt. \tag{39}$$

The immediate problem is to show that for Case I $\{w_\alpha/\lambda_\alpha\}$ is of finite norm. We introduce a transformation B such that

$$\int\int T(x, \xi)B[f(x, s)]\overline{T}(\eta, s)dxds = A[T(x, \xi)f(x, s)\overline{T}(\eta, s)], \tag{40}$$

A denoting the transformation defined by (27). The transformed function $B[g(s)\int L(x, y)f(y)dy]$ exists and is a continuous function, for it may be

expressed in the following form by series that are uniformly convergent,

$$B\left[g\int Lf\right] = \sum_{ik}\int K\varphi_i\int f\psi_i\frac{\varphi_k}{\alpha_i^2+\beta_k^2}\int g\bar{\psi}_k + \sum_{ik}\frac{\varphi_i}{\alpha_i^2}\int f\psi_i\frac{\beta_k^2}{\alpha_i^2+\beta_k^2}\int g\bar{\psi}_k\int M\varphi_k. \tag{41}$$

In the second member of (41), $\int K\varphi_i = \int\int \boldsymbol{K}T\varphi_i = \int \boldsymbol{K}\psi_i$, and $\{\int \boldsymbol{K}\psi_i\}$ is of finite norm. Likewise $\{\int f\psi_i\}$, $\{\int g\bar{\psi}_k\}$, and $\{\int M\bar{\varphi}_k\} = \{\int \boldsymbol{M}\bar{\psi}_k\}$ are of finite norm. If we consider the kernel N as a function symmetrizable on the right by the function $\boldsymbol{N}$, which is of positive type, by Mercer's results* we have the expansion

$$\int N(s, \eta)\boldsymbol{N}(\eta, t)d\eta = \sum_n \frac{\varphi_n(s)\bar{\varphi}_n(t)}{\beta_n^4},$$

and the uniform convergence of the series on the right follows from the uniform convergence of the series in Mercer's expansion for $\boldsymbol{N}(s, t)$.† We conclude, therefore, that $\{\bar{\varphi}_k/\beta_k^2\}$ is of finite norm. Similarly, in the expansion

$$\int L(x, \xi)\boldsymbol{L}(\xi, y)d\xi = \sum_n \frac{\varphi_n(x)\varphi_n(y)}{\alpha_n^4}$$

the series is uniformly convergent, and $\{\varphi_i(x)/\alpha_i^2\}$ is of finite norm. Hence, the two series in (41) are uniformly convergent and represent a continuous function. The transformed function $B[f(x)\int N(s, t)g(t)dt]$ exists also, and is continuous. We have then

$$\int\int T(x, p)B[g(s)\int L(x, y)f(y)dy + f(x)\int N(s, t)g(t)dt]\bar{T}(s, q)dxds$$
$$= A[g^*(q)\int L(y, p)f^*(y)dy + f^*(p)\int N(t, q)g^*(t)dt].$$

When $f = u_\alpha$ and $g = v_\alpha$, this becomes

$$\int\int T(x, p)B[v_\alpha(s)\int L(x, y)u_\alpha(y)dy + u_\alpha(x)\int N(s, t)v_\alpha(t)dt]\bar{T}(s, q)dxds = A[w_\alpha^*(p, q)].$$

* Mercer, loc. cit., p. 409. For a kernel k symmetrizable on the right by γ', Mercer gives the expansion

$$\int_a^b k(s, x)\gamma'(x, t)dx = \sum_n \frac{\varphi_n(s)\varphi_n(t)}{\lambda_n\mu_n'},$$

where $\{\varphi_n, \psi_n\}$ is the complete biorthogonal system defined by $\varphi_n = \lambda_n\int k\varphi_n$ and $\psi_n = \lambda_n\int\psi_n k$, and

$$\mu_n' = \frac{1}{\int\int\gamma'(s, t)\psi_n(s)\psi_n(t)dsdt}.$$

† Mercer, loc. cit., p. 409. In Mercer's notation,

$$\boldsymbol{N} = \gamma'(s, t) = \sum_n \frac{\varphi_n(s)\varphi_n(t)}{\mu_n'} + \sum_n \frac{\xi_n'(s)\xi_n'(t)}{\nu_n'}.$$

Substituting for $A[w_\alpha^*(p, q)]$ from (30), and reducing, we obtain

$$(42) \qquad B[w_\alpha(x, s)] = \frac{w_\alpha(x, s)}{\lambda_\alpha}.$$

By a process similar to that used in passing from (29) to (32), the following inequality is obtained from (41),

$$(43) \qquad 2\left|B\left[g(s)\int L(x, y)f(y)dy\right]\right| \leq \sum_{i,k}\frac{[\int f(\xi)\psi_i(\xi)d\xi]^2[\int g(\eta)\overline{\psi}_k(\eta)d\eta]^2}{\alpha_i^2} + \Phi_1^2(x, s),$$

where

$$\Phi_1^2(x, s) = 2\sum_{i,k}\left[\left(\int K(x, y)\varphi_i(y)dy\right)^2\frac{\overline{\varphi}_k^2(s)}{\alpha_i^2+\beta_k^2} + \frac{\varphi_i^2(x)}{\alpha_i^2}\left(\int M(s, t)\overline{\varphi}_k(t)dt\right)^2\right],$$

a continuous function, as the series on the right is uniformly convergent. The uniform convergence of $\sum_i(\int K\varphi_i)^2$ and $\sum_k(\int M\overline{\varphi}_k)^2$ for the special kernels under consideration has already been noted; the uniform convergence of $\sum_i[\varphi_i^2(x)/\alpha_i^2]$ and $\sum_k[\overline{\varphi}_k^2(s)/\beta_k^2]$ results from Mercer's theorem,* if we regard N and L as symmetrizable on the right by $\boldsymbol{N}$ and $\boldsymbol{L}$, respectively. The inequality (43) may be put into the form

$$(44) \qquad 2|B[g\int Lf] \leq \int\int fL^*f\int gg^* + \Phi_1^2,$$

and for $B[f\int Ng]$ there is a corresponding inequality,

$$(45) \qquad 2|B[f\int Ng] \leq \int\int gN^*g\int ff^* + \Psi_1^2,$$

* Mercer, loc. cit., p. 409. The expansion referred to is

$$\gamma'(s, t) = \frac{\sum_n \varphi_n(s)\varphi_n(t)}{\mu_n'} + \sum_n\frac{\xi_n'(s)\xi_n'(t)}{v_n'},$$

in which each series is uniformly convergent when $\gamma'(s, s)$ is a continuous function of s in the interval considered. In the notation here used, the first series in the corresponding expansion for $\boldsymbol{L}$ is

$$\sum_n\frac{\varphi_n(x)\varphi_n(y)}{\alpha_n^2}.$$

Hence $\sum_n \varphi_n^2(x)/\alpha_n^2$ is uniformly convergent. Similarly, from the expansion for N, we have the uniform convergence of $\sum_n\overline{\varphi}_n(s)\overline{\varphi}_n(t)/\beta_k^2$.

in which Ψ_1^2 represents a continuous function of x and s. From (44) and (45) there results the relation

$$
\begin{aligned}
(46)\quad & \Phi_1^2(x, s) - 2B[\int L(x, y)f(y, s)dy] \\
& \quad + \int\int f(x, s)L^*(x, y)f(y, \eta)dxdy \int\int \overline{T}(\eta, s)d\eta ds \\
& \quad + \Psi_1^2(x, s) - 2B[\int N(s, t)f(t, x)dt] \\
& \quad + \int\int f(x, s)N^*(s, t)f(\xi, t)dsdt \int\int T(\xi, x)d\xi dx \geq 0.
\end{aligned}
$$

If

$$
(47)\qquad f(y, t) = \sum_{\alpha=1}^{n} B_\alpha u_\alpha(y)v_\alpha(t)
$$

and

$$
B_\alpha = \frac{w_\alpha(x, s)}{\lambda_\alpha},
$$

then by (42)

$$
B_\alpha = B[w_\alpha(x, s)].
$$

With the substitution for f of the function given by (47), the inequality (46) reduces to

$$
\Phi_1^2(x, s) - 2\sum_\alpha B_\alpha^2 + \sum_\alpha B_\alpha^2 + \Psi_1^2(x, s) \geq 0
$$

or

$$
\sum_{\alpha=1}^{n} B_\alpha^2 \leq \Phi_1^2 + \Psi_1^2.
$$

Therefore $\{w_\alpha(x, s)/\lambda_\alpha\}$ is of finite norm. From this result and the fact that $\{f_i\}$ is of finite norm, we conclude that the series

$$
\sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha} \int\int f(y, t)w_\alpha^*(y, t)dydt
$$

is uniformly convergent. The following theorem has then been proved:

THEOREM 3: *If the kernels of equations* (1) *have the form* $K = \int \mathbf{K}T$, $L = \int \mathbf{L}T$, $M = \int \mathbf{M}\overline{T}$, $N = \int \mathbf{N}\overline{T}$, *where* $\mathbf{L}$, $\mathbf{N}$, T, *and* $\overline{T}$ *are symmetric and positive definite, and* $\mathbf{K}$ *and* $\mathbf{M}$ *are symmetric, a function* $g(x, s)$ *which has the form*

$$
g(x, s) = \int\int [K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)]dydt,
$$

$f(y, t)$ *being any continuous function, may be expanded into the uniformly convergent series*

$$
g(x, s) = \sum_\alpha w_\alpha(x, s) \int\int g(\xi, \eta)u_\alpha^*(\xi)v_\alpha^*(\eta)d\xi d\eta,
$$

where w_α is the function defined by (18) *and u_α^*, v_α^* are the solutions of the adjoint system* (2).

Expansion Theorems for Case II.—Before taking up the question of the expansion of arbitrary functions, we note certain reductions that are a consequence of the special form of the kernels. The equations now considered are

$$(48)\qquad \begin{aligned} u(x) &= \lambda\int a(x)T(x, y)u(y)dy + \mu\int b(x)T(x, y)u(y)dy, \\ v(s) &= \lambda\int \bar{a}(s)\overline{T}(s, t)v(t)dt - \mu\int \bar{b}(s)\overline{T}(s, t)v(t)dt, \end{aligned}$$

which may be written

$$(49)\qquad \begin{aligned} u(x) &= [\lambda a(x) + \mu b(x)]u^*(x), \\ v(s) &= [\lambda\bar{a}(s) - \mu\bar{b}(s)]v^*(s). \end{aligned}$$

Substituting from (49) in $w_\alpha(x, s)$, we obtain

$$(50)\qquad \begin{aligned} w_\alpha(x, s) &= \lambda_\alpha[a(x)\bar{b}(s) + b(x)\bar{a}(s)]u_\alpha^*(x)v_\alpha^*(s) \\ &= \lambda_\alpha F(x, s)u_\alpha^*(x)v_\alpha^*(s), \end{aligned}$$

where

$$F(x, s) = a(x)\bar{b}(s) + b(x)\bar{a}(s).$$

The equations (8) which define the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$ give the relations

$$\varphi_i(x) = \alpha_i^2 b(x)\psi_i(x), \qquad \overline{\varphi}_k(s) = \beta_k^2\bar{b}(s)\overline{\psi}_k(s).$$

It follows that

$$\alpha_i^2\int b\psi_i\psi_j = \delta_{ij}$$

and therefore

$$\{\chi_i\} = \{\alpha_i\sqrt{b}\psi_i\}$$

is a closed normalized orthogonal system of functions. Similarly

$$\{\overline{\chi}_k\} = \{\beta_k\sqrt{\bar{b}}\overline{\psi}_k\}$$

is a closed normalized orthogonal system. We pass from the biorthogonal systems to the orthogonal by means of the relations

$$\varphi_i = \alpha_i\sqrt{b}\chi_i, \qquad \psi_i = \frac{1}{\alpha_i}\frac{\chi_i}{\sqrt{b}},$$

$$\overline{\varphi}_k = \beta_k\sqrt{\bar{b}}\overline{\chi}_k, \qquad \overline{\psi}_k = \frac{1}{\beta_k}\frac{\chi_k}{\sqrt{\bar{b}}}.$$

From (8)

$$\chi_i(x) = \alpha_i^2 \int \sqrt{b(x)}\, T(x, \xi) \sqrt{b(\xi)}\, \chi_i(\xi) d\xi,$$

$$\bar{\chi}_k(s) = \beta_k^2 \int \sqrt{\bar{b}(s)}\, \bar{T}(s, \eta) \sqrt{\bar{b}(\eta)}\, \bar{\chi}_k(\eta) d\eta.$$

It follows from (14) and (3) that

$$\int\int u_\alpha^*(x) v_\alpha^*(s) \cdot \lambda_\beta F(x, s) u_\beta^*(x) v_\beta^*(s) dx ds = \delta_{\alpha\beta},$$

hence

$$\lambda_\alpha \int\int F(x, s) u_\alpha^{*2}(x) v_\alpha^{*2}(s) dx ds = 1,$$

which shows that $\lambda_\alpha > 0$ if $F(x, s) > 0$. Therefore, when $F(x, s) > 0$ and $\lambda_\alpha > 0$, the functions $\{\sqrt{\lambda_\alpha}\sqrt{F(x, s)}\, u_\alpha^*(x) v_\alpha^*(s)\} = \{\zeta_\alpha\}$ form an orthogonal system.

We now proceed to prove that the series (39) is uniformly convergent when the kernels are of the type under consideration and $F(x, s) > 0$. As

$$\sum_\alpha{}' \frac{w_\alpha(x, s)}{\lambda_\alpha} \int\int f(y, t) w_\alpha^*(y, t) dy dt$$
$$= \sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha^{3/2}} \cdot \sqrt{\lambda_\alpha} \int\int f(y, t) w_\alpha^*(y, t) dy dt,$$

the series is uniformly convergent if $\{w_\alpha(\xi, \eta)/\lambda_\alpha^{3/2}\}$ and $\{\sqrt{\lambda_\alpha}\int\int f w_\alpha^*\}$ are of finite norm. Let B denote the transformation defined by (40). In

$$B[f(x, s)] = \sum_{ik} \int K\varphi_i \int\int \psi_i f \bar{\psi}_k \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} \bar{\varphi}_k$$
$$+ \sum_{ik} \frac{\varphi_i}{\alpha_i^2} \int\int \psi_i f \bar{\psi}_k \int M\bar{\varphi}_k \cdot \frac{\beta_k^2}{\alpha_i^2 + \beta_k^2},$$

substitute aT and $\bar{a}\bar{T}$ for K and M, respectively, and change from the biorthogonal functions $\{\varphi_i, \psi_i\}$, $\{\bar{\varphi}_k, \bar{\psi}_k\}$ to the orthogonal functions $\{\chi_i\}$, $\{\bar{\chi}_k\}$, thus obtaining

$$B[f(x, s)] = \sum_{ik} \frac{\chi_i(\xi) f(\xi, \eta) \bar{\chi}_k(\eta) d\xi d\eta}{\sqrt{b(\xi)}\sqrt{\bar{b}(\eta)}} \cdot \frac{F(x, s)}{\sqrt{b(x)}\sqrt{\bar{b}(s)}} \cdot \frac{\chi_i(x)\bar{\chi}_k(s)}{\alpha_i^2 + \beta_k^2}.$$

Then

$$2|B[f(x, s)]| \leq \int\int \frac{f^2(\xi, \eta) d\xi d\eta}{b(\xi)\bar{b}(\eta)} + \Pi^2(x, s),$$

where

$$\Pi^2(x, s) = \frac{F^2(x, s)}{b(x)\bar{b}(s)} \cdot \sum_{ik} \frac{\chi_i^2(x)}{\alpha_i^2} \cdot \frac{\bar{\chi}_k^2(s)}{\beta_k^2}.$$

Since

$$\int\int f(x)\sqrt{b(x)}\,T(x, y)\sqrt{b(y)}f(y)dxdy \geq 0$$

the series $\sum_i [\chi_i^2(x)/\alpha_i^2]$ is uniformly convergent.* The series $\sum_k [\bar{\chi}_k^2(s)/\beta_k^2]$ is also uniformly convergent, hence we conclude that $\Pi^2(x, s)$ is a continuous function. Let us assume that $F(x, s)$ satisfies the condition $0 < F(x, s) < b(x)\bar{b}(s)$, a condition that may be imposed without limiting the problem. Then

$$\int\int \frac{f^2(\xi, \eta)}{b(\xi)\bar{b}(\eta)}d\xi d\eta \leq \int\int \frac{f^2(\xi, \eta)}{F(\xi, \eta)}d\xi d\eta,$$

and it follows that

$$2|B[f(x, s)]| \leq \int\int \frac{f^2(\xi, \eta)}{F(\xi, \eta)}d\xi d\eta + \Pi^2(x, s). \tag{51}$$

If

$$f(x, s) = \sum_{\alpha=1}^{n} C_\alpha \sqrt{\lambda_\alpha}F(x, s)u_\alpha^*(x)v_\alpha^*(s),$$

where

$$C_\alpha = \frac{w_\alpha(y, t)}{\lambda_\alpha^{3/2}} = B\left[\frac{w_\alpha(y, t)}{\sqrt{\lambda_\alpha}}\right]$$

the inequality (51) becomes

$$\Pi^2(x, s) - 2\sum_{\alpha=1}^{n} C_\alpha B[\sqrt{\lambda_\alpha}F(x, s)u_\alpha^*(x)v_\alpha^*(s)]$$

$$+ \sum_{\alpha,\beta=1}^{n} C_\alpha C_\beta \int\int \sqrt{\lambda_\alpha}\sqrt{\lambda_\beta}F(\xi, \eta)u_\alpha^*(\xi)v_\alpha^*(\eta)u_\beta^*(\xi)v_\beta^*(\eta)d\xi d\eta \geq 0.$$

This reduces to

$$\Pi^2(x, s) - 2\sum_{\alpha=1}^{n} C_\alpha \cdot B\left[\frac{w_\alpha(x, s)}{\sqrt{\lambda_\alpha}}\right] + \sum_{\alpha=1}^{n} C_\alpha^2 \geq 0$$

or

$$\sum_{\alpha=1}^{n} C_\alpha^2 \leq \Pi^2(x, s).$$

Hence $\{w_\alpha(y, t)/\lambda_\alpha^{3/2}\}$ is of finite norm. To show $\{\sqrt{\lambda_\alpha}\int\int f(x, s)w_\alpha^*(x, s)dxds\}$ of finite norm, substitute for $w_\alpha^*(x, s)$ from (15) and exhibit ζ_α as a factor

* Mercer, loc. cit., p. 407.

of each term, thus obtaining

$$\sqrt{\lambda_\alpha}\int\int f(x,\,s)w_\alpha^*(x,\,s)dxds$$

$$=\int\int\int\frac{f(x,\,s)N(s,\,t)}{\sqrt{F(x,\,t)}}ds\,\sqrt{\lambda_\alpha}\,\sqrt{F(x,\,t)}u_\alpha^*(x)v_\alpha^*(t)dxdt$$

$$+\int\int\int\frac{f(x,\,s)L(x,\,y)}{\sqrt{F(y,\,s)}}dx\,\sqrt{\lambda_\alpha}\,\sqrt{F(s,\,y)}u_\alpha^*(y)v_\alpha^*(s)dyds$$

$$=\int\int h(x,\,t)\zeta_\alpha(x,\,t)dxdt,$$

where

$$h(x,\,t)=\int\frac{f(x,\,s)N(s,\,t)}{\sqrt{F(x,\,t)}}ds+\int\frac{f(x,\,s)L(x,\,y)}{\sqrt{F(y,\,s)}}dx.$$

But

$$\sum_\alpha\left[\int\int h(x,\,t)\zeta_\alpha(x,\,t)dxdt\right]^2\leq\int\int[h(x,\,t)]^2dxdt;$$

therefore

$$\sum_\alpha\left[\sqrt{\lambda_\alpha}\int\int f(x,\,s)w_\alpha^*(x,\,s)dxds\right]^2\leq 2\int\int[h(x,\,t)]^2dxdt;$$

that is, $\{\sqrt{\lambda_\alpha}\int\int f(x,\,s)w_\alpha^*(x,\,s)dxds\}$ is of finite norm. Since each of the sequences is of finite norm, the series (47) is uniformly convergent.

The function to which the series converges, $\int\int[K(x,\,y)f(y,\,t)N(s,\,t) + L(x,y)f(y,t)M(s,t)]dydt$, takes the form $F(x,s)\int\int T(x,y)f(y,t)\overline{T}(t,s)dydt$ for the special kernels considered. Hence, with the assumption $F>0$, we have obtained the expansion

$$F(x,\,s)\int\int T(x,\,y)f(y,\,t)\overline{T}(t,\,s)dydt=\sum_\alpha\frac{w_\alpha(x,\,s)}{\lambda_\alpha}\int\int f(\xi,\,\eta)w_\alpha^*(\xi,\,\eta)d\xi d\eta.$$

Because of (50), this reduces to

$$\int\int T(x,\,y)f(y,\,t)\overline{T}(t,\,s)dydt=\sum_\alpha u_\alpha^*(x)v_\alpha^*(s)\int\int f(\xi,\,\eta)w_\alpha^*(\xi,\,\eta)d\xi d\eta.$$

These results give the following theorem:

THEOREM 4: *If the kernels of system* (1) *have the form* $K(x,y)=a(x)T(x,y)$, $L(x,\,y)=b(x)T(x,\,y)$, $M(s,\,t)=\bar{a}(s)\overline{T}(s,\,t)$, $N(s,\,t)=\bar{b}(s)\overline{T}(s,\,t)$, *where* a, b, $\bar{a}$, $\bar{b}$ *are continuous functions,* $b>0$, $\bar{b}>0$, $a\bar{b}+\bar{a}b>0$ *and* T, $\overline{T}$ *are symmetric positive definite kernels, a function* $g(x,\,s)$ *which is expressible as* $\int\int T(x,\,y)f(y,\,t)\overline{T}(t,\,s)dydt$, $f(y,\,t)$ *denoting a continuous function of* y

and t, may be expanded into the uniformly convergent series

$$g(x, s) = \sum_{\alpha} u_{\alpha}^{*}(x)v_{\alpha}^{*}(s) \int\int f(\xi, \eta)w_{\alpha}^{*}(\xi, \eta)d\xi d\eta$$
$$= \sum_{\alpha} u_{\alpha}^{*}(x)v_{\alpha}^{*}(s) \int\int g(y, t)w_{\alpha}(y, t)dydt,$$

where u_{α}^{*}, v_{α}^{*} *are the solutions of the adjoint system* (2) *and* w_{α}^{*} *and* w_{α} *are the functions defined by* (15) *and* (18), *respectively.*

We now show without any assumption as to the positive character of $F(x, s)$ that the series

$$\sum_{\alpha} \frac{w_{\alpha}(x, s)}{\lambda_{\alpha}^{2}} \int\int f(y, t)w_{\alpha}^{*}(y, t)dydt$$

is uniformly convergent. Using again the transformation B, we find that the transformed function $B[B[g(s)\int L(x, y)f(y)dy]]$ has the following form:

$$B[B[g\int Lf]] = \sum_{ik}\sum_{jl} \frac{\int f(y)\psi_i(y)dy \int g(\eta)\bar{\psi}_l(\eta)d\eta}{\alpha_j}$$
$$\times \int\int \psi_i(\xi)\bar{\psi}_k(\eta)\frac{F(\xi, \eta)}{b(\xi)\bar{b}(\eta)}\cdot\frac{\varphi_j(\xi)\bar{\varphi}_l(\eta)}{\alpha_j(\alpha_j^2+\beta_l^2)}d\xi d\eta\frac{F(x, s)}{b(x)\bar{b}(s)}\cdot\frac{\varphi_i(x)\bar{\varphi}_k(s)}{\alpha_i^2+\beta_k^2}.$$

Upon the introduction of the orthogonal functions $\{\chi_i\}$ and $\{\bar{\chi}_k\}$ this becomes

$$B[B[g\int Lf]] = \sum_{jl} \frac{\int f(y)\psi_i(y)dy \int g(\eta)\bar{\psi}_l(\eta)d\eta}{\alpha_j}$$

$$\times\left[\sum_{i}\frac{\int \chi_i(\xi)\frac{a(\xi)}{b(\xi)}\chi_j(\xi)d\xi}{\alpha_j^2+\beta_l^2}\sqrt{b(x)\bar{b}(s)}\frac{\beta_l\chi_i(x)\bar{\chi}_l(s)}{\alpha_i^2+\beta_l^2}\right. \tag{52}$$
$$\left.+\sum_{k}\frac{\int \bar{\chi}_k(\eta)\frac{\bar{a}(\eta)}{\bar{b}(\eta)}\bar{\chi}_l(\eta)}{\alpha_j^2+\beta_l^2}\sqrt{b(x)\bar{b}(s)}\frac{\beta_l\chi_j(x)\bar{\chi}_k(s)}{\alpha_j^2+\beta_k^2}\right].$$

By a series of operations similar to those employed in passing from (31) to (33), there result from equation (52) and the corresponding equation for $B[B[g\int Lf]]$ the inequalities

$$2|B[B[g\int Lf]]| \leq \int\int fL^{*}f\int\int gg^{*} + \Pi_1^2, \tag{53}$$

$$2|B[B[f\int Ng]]| \leq \int\int gN^{*}g\int ff^{*} + \Pi_2^2, \tag{54}$$

Π_1^2 and Π_2^2 denoting continuous functions. As a consequence it follows that

$$\Pi_1^2(x, s) + \int\int f(x, s)L^{*}(x, y)f(y, s)dxds\int\int \bar{T}(\eta, s)d\eta ds$$
$$+ \Pi_2^2(x, s) + \int\int f(x, s)N^{*}(s, t)f(\xi, t)dsdt\int\int T(\xi, x)d\xi dx$$
$$- 2B[B[L(x, y)f(y, s)dy]] - 2B[B[N(s, t)f(t, x)dt]] \geq 0.$$

Let

$$f(y, t) = \sum_{\alpha=1}^{n} B_\alpha u_\alpha(y) v_\alpha(t),$$

where

$$B_\alpha = \frac{w_\alpha(x, s)}{\lambda_\alpha^2}.$$

The substitution of this value of f reduces the preceding inequality to

$$\Pi_1^2(x, s) + \Pi_2^2(x, s) + \sum_{\alpha=1}^{n} B_\alpha^2 - 2 \sum_{\alpha=1}^{n} B_\alpha^2 \geq 0,$$

or

$$\sum_{\alpha=1}^{n} B_\alpha^2 \leq \Pi_1^2 + \Pi_2^2.$$

Hence $\{w_\alpha/\lambda_\alpha^2\}$ is of finite norm and the series

$$\sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha^2} \int\int f(y, t) w_\alpha^*(y, t) dy dt$$

is uniformly convergent.

The next step is to determine the form of the function represented by the expansion. Equation (21) gives the relation

$$a_{ikjl} = \sum_\alpha \frac{c_{ik}^2 x_{\alpha ik}^* \cdot c_{jl}^2 x_{\alpha jl}^*}{\lambda_\alpha},$$

which, in turn, gives

$$\sum_{j_1 l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot a_{j_1l_1jl} = \sum_\alpha \frac{\sum_{j_1l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot c_{j_1l_1}^2 x_{\alpha j_1l_1}^* \cdot c_{jl}^2 x_{\alpha jl}^*}{\lambda_\alpha}.$$

By (26), this reduces to

$$\sum_{j_1l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot a_{j_1l_1jl} = \sum_\alpha \frac{c_{ik}^2 x_{\alpha ik} \cdot c_{jl}^2 x_{\alpha jl}}{\lambda_\alpha^2}$$

or

$$\sum_{j_1l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot a_{j_1l_1jl} = \sum_\alpha \frac{w_{\alpha ik} w_{\alpha jl}}{\lambda_\alpha^2},$$

and this by means of the relation

$$w_{\alpha ik} = \int\int \psi_i \bar{\psi}_k w_\alpha = \int\int \varphi_i \bar{\varphi}_k w_\alpha^*$$

and the definition of the transformation B gives the equality

$$B\left[\int\int \{K(x, y) f(y, t) N(s, t) + L(x, y) f(y, t) M(s, t)\} dy dt\right]$$
$$= \sum_\alpha \frac{w_\alpha(xs)}{\lambda_\alpha^2} \int\int f(y, t) w_\alpha^*(y, t) dy dt.$$

This completes the proof of

THEOREM 5: *Any continuous function $h(x, s)$ that is expressible in the form*

$$h(x, s) = B[\int\int\{K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)\}dydt],$$

where K, L, M, and N are kernels of the type of Case II and $f(y, t)$ is a continuous function of y and t, may be expanded into the uniformly convergent series

$$\begin{aligned} h(x, s) &= \sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha^2} \int\int f(y, t)w_\alpha^*(y, t)dydt \\ &= \sum_\alpha w_\alpha(x, s) \int\int h(y, t)u_\alpha^*(y)v_\alpha^*(t)dydt \\ &= F(x, s) \sum_\alpha u_\alpha^*(x)v_\alpha^*(s)\cdot\lambda_\alpha \int\int h(y, t)u_\alpha^*(y)v_\alpha^*(t)dydt, \end{aligned}$$

where w_α and w_α^* are the functions defined by (18) *and* (15) *respectively, and u_α^*, v_α^* are the solutions of the adjoint system* (2) *with the kernels of Case II.*

Let $T(x, y)$ and $\overline{T}(s, t)$ denote the Green's functions that vanish at the ends of the intervals considered, of the differential expressions $d\left(p\frac{du^*}{dx}\right) / dx$ and $d\left(\pi\frac{dv^*}{ds}\right) / ds$, respectively, where $p > 0$ and $\pi > 0$. The system of differential equations (b) (§ 1) with the given boundary conditions is equivalent to a system of integral equations of the form

$$\begin{aligned} u^*(x) &= \lambda \int_a^b T(x, y)a(y)u^*(y)dy + \mu \int_a^b T(x, y)b(y)u^*(y)dy, \\ v^*(s) &= \lambda \int_a^b \overline{T}(s, t)\bar{a}(t)v^*(t)dt - \mu \int_c^d \overline{T}(s, t)\bar{b}(t)v^*(t)dt, \end{aligned}$$

a, b, $\bar{a}$, $\bar{b}$ denoting analytic functions, and this is exactly the adjoint system of (48). By Theorem 4 any continuous function having continuous first and second derivatives and satisfying the given boundary conditions may be expanded into a uniformly convergent series in terms of the solutions u_α^*, v_α^* of the differential equations, if $F(x, s) > 0$.

If $F(x, s)$ is not everywhere positive, it follows from Theorem 5 that every function $g(x, s)$ continuous and with continuous derivatives of the first four orders, satisfying certain boundary conditions, may be expanded into a uniformly convergent series in terms of $u_\alpha^* v_\alpha^*$. For when $g(x, s)$ is subject to the above conditions, there exists a function $k(x, s)$ continuous and with continuous first and second derivatives, satisfying certain boundary conditions, such that

$$g(x, s) = \sqrt{b(x)}\int T(x, y)b(y)dy\cdot k(x, s)\sqrt{\bar{b}(s)}\int \overline{T}(s, t)\bar{b}(t)dt.$$

Hence

$$\int\int g(x, s)\chi_i(x)\overline{\chi}_k(s)dxds = \frac{\int\int \chi_i(x)k(x, s)\overline{\chi}_k(s)dxds}{\alpha_i^2\beta_k^2}.$$

Further a function $f(y, t)$ can be found such that

$$\begin{aligned}\int\int T(\xi, y)f(y, t)\overline{T}(\eta, t)dydt & \\ = b(\xi)\sqrt{\overline{b}(\eta)}\int \overline{T}(\xi, \eta_1)b(\eta_1)k(\eta, \eta_1)d\eta_1 & \\ + b(\xi)\overline{b}(\eta)\int T(\eta_1, \eta)\sqrt{\overline{b}(\eta_1)}k(\xi, \eta_1)d\eta_1 &\end{aligned}$$

and therefore

$$\begin{aligned}\int\int\int\int T(\xi, y)f(y, t)\overline{T}(\eta, t)dydt\cdot\frac{F(\xi, \eta)}{\sqrt{b(\xi)}\sqrt{\overline{b}(\eta)}}\cdot\frac{\chi_i(\xi)\overline{\chi}_k(\eta)}{\alpha_i^2+\beta_k^2} & \\ = \int\int \chi_i(\xi)k(\xi, \eta)\overline{\chi}_k(\eta)d\xi d\eta\cdot\frac{\alpha_i^2+\beta_k^2}{\alpha_i^2\beta_k^2}\cdot\frac{1}{\alpha_i^2+\beta_k^2} & \\ = \int\int g(\xi, \eta)\chi_i(\xi)\overline{\chi}_k(\eta)d\xi d\eta. &\end{aligned}$$

It follows that

$$\begin{aligned}B\left[\int\int \{K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)\}dydt\right. & \\ = g(x, s)\frac{F(x, s)}{\sqrt{b(x)\overline{b}(s)}}; &\end{aligned}$$

that is, the function g is expressible in the form specified for the function h in Theorem 5.

Thus Hilbert's results for the expansion of an arbitrary function of two variables in terms of the solutions of the differential equations (b) are obtained as a special case of the results stated in Theorems 4 and 5.

THE ASYMPTOTIC EXPANSION OF THE FUNCTIONS $W_{k,m}(z)$ OF WHITTAKER.

By F. H. Murray.

The asymptotic expansion of the functions $W_{k,m}(z)$ has been given* for any sector $|\arg z| < \pi - \epsilon$ where $\epsilon > 0$.

In many applications it is convenient to have also the expansion in a sector including the negative half of the real axis; in this paper it will be shown that if the parameters k and m satisfy certain inequalities, the expansion given by Whittaker remains valid in such a sector. This result is applied in a study of the "croissance" of the solutions of a class of linear differential equations of the second order, forming an extension of an earlier paper by the writer.

1. If $R(k - \frac{1}{2} - m) \leq 0$, the function $W_{k,m}(z)$ is defined by the formula

$$W_{k,m}(z) = \frac{e^{-(z/2)}z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty t^{-k-1/2+m}\left(1 + \frac{t}{z}\right)^{k-1/2+m} e^{-t}dt \tag{1}$$

$$(-\pi < \arg z < \pi)$$

and satisfies the differential equation

$$\frac{d^2W}{dz^2} + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right\} W = 0. \tag{2}$$

If z is not real, $W_{k,m}(z)$ and $W_{-k,m}(-z)$ are linearly independent solutions of (2); if z is real, the values $W_{k,m}(z + io)$ or $W_{k,m}(z - io)$ can be taken. When z is not real, the asymptotic expansion for $W_{k,m}(z)$ is known; it will be shown that if k, m are real, and

$$-1 < k - \tfrac{1}{2} + m < 0, \tag{3}$$

this expansion remains valid when z approaches a point on the negative real axis.

If z is not real, and $\lambda = k - \frac{1}{2} + m$,

$$\begin{aligned} \left(1 + \frac{t}{z}\right)^\lambda &= 1 + \lambda\frac{t}{z} + \frac{\lambda(\lambda - 1)}{2!}\left(\frac{t}{z}\right)^2 + \cdots \\ &+ \frac{\lambda(\lambda - 1)\cdots(\lambda - n + 1)}{n!}\left(\frac{t}{z}\right)^n \\ &+ \frac{\lambda(\lambda - 1)\cdots(\lambda - n)}{n!}\left(1 + \frac{t}{z}\right)^\lambda \int_0^{t/z} u^n(1 + u)^{-1-\lambda}du, \end{aligned} \tag{4}$$

* *Bull. Amer. Math. Soc.*, Vol. X. Whittaker and Watson, "Modern Analysis," Chap. XVI, 2d or 3d edition.

where for convenience a straight line path of integration will be chosen. As in (1), that branch of the function $[1 + (t/z)]^{\lambda}$ is chosen which is real and positive when z is real and positive. If the right-hand member of (4) is substituted in (1) and the terms integrated separately, the following expansion is obtained:

$$
\begin{aligned}
W_{k,\,m}(z) = e^{-(z/2)}z^k \Bigg\{ 1 &+ \frac{m^2 - (k-\frac{1}{2})^2}{1!\,z} \\
(5)\qquad &+ \frac{\{m^2-(k-\frac{1}{2})^2\}\{m^2-(k-\frac{3}{2})^2\}}{2!\,z^2} + \cdots \\
&+ \frac{\{m^2-(k-\frac{1}{2})^2\}\cdots\{m^2-(k-n+\frac{1}{2})^2\}}{n!\,z^n} + R_n(z) \Bigg\},
\end{aligned}
$$

in which

$$
(6)\qquad R_n(z) = \frac{\lambda(\lambda-1)\cdots(\lambda-n)}{n!\,\Gamma(-k+\frac{1}{2}+m)}\int_0^{\infty} t^{\lambda-2k}e^{-t}\left(1+\frac{t}{z}\right)^{\lambda}\left[\int_0^{t/z} u^n(1+u)^{-1-\lambda}du\right]dt,
$$

$$
n-k-\tfrac{1}{2}+m > 0.
$$

It remains to discuss the remainder term when z is in the neighborhood of a point on the negative half of the real axis. If t is real, and $z = -z'$,

$$
z' = x+iy, \qquad y \neq 0,
$$

$$
(7)\qquad R\left(\frac{t}{z'}\right) = \frac{tx}{|z'|^2} = \frac{t}{t_1}
$$

if

$$
t_1 = \frac{|z'|^2}{x} = |z'|\sec\alpha, \qquad \alpha = \arctan\frac{y}{x}\,.
$$

Also

$$
\int_0^{t/z'} u^n(1+u)^{-\lambda-1}du = (-1)^{n-1}\int_0^{t/z'} u^n(1-u)^{-\lambda-1}du.
$$

Substituting in (6),

$$
(8)\qquad R_n(-z') = (-1)^{n-1}\frac{\lambda(\lambda-1)\cdots(\lambda-n)}{n!\,\Gamma(-k+\frac{1}{2}+m)}[I_1+I_2],
$$

$$
(9)\qquad
\begin{aligned}
I_1 &= \int_0^{t_1} t^{\lambda-2k}e^{-t}\left(1-\frac{t}{z'}\right)^{\lambda}\left[\int_0^{t/z'} u^n(1-u)^{-\lambda-1}du\right]dt,\\
I_2 &= \int_{t_1}^{\infty} t^{\lambda-2k}e^{-t}\left(1-\frac{t}{z'}\right)^{\lambda}\left[\int_0^{t/z'} u^n(1-u)^{-\lambda-1}du\right]dt.
\end{aligned}
$$

If a straight line path of integration is chosen, and $u = \bar{x} + i\bar{y}$, then since $-\lambda - 1 < 0$,

$$
(10)\qquad \begin{gathered}
|du| = \sec\alpha d\bar{x}, \qquad |1-u| \geq |1-\bar{x}|, \\
|(1-u)^{-\lambda-1}| \leq |(1-\bar{x})^{-\lambda-1}|, \\
\left|1-\frac{t}{z'}\right| \geq \left|1-\frac{t}{t_1}\right|, \qquad \left|\left(1-\frac{t}{z'}\right)^{\lambda}\right| \leq \left|\left(1-\frac{t}{t_1}\right)^{\lambda}\right|.
\end{gathered}
$$

Consequently* if $t < t_1$,

$$
(11)\qquad \begin{aligned}
\left|\int_0^{t/z'} u^n(1-u)^{-\lambda-1}du\right| &\leq \left|\frac{t}{z'}\right|^n \sec\alpha \int_0^{t/t_1}(1-\bar{x})^{-\lambda-1}d\bar{x} \\
&\leq \left|\frac{\sec\alpha}{\lambda}\left(\frac{t}{z'}\right)^n\right| \cdot \left|1-\left(1-\frac{t}{t_1}\right)^{-\lambda}\right|,
\end{aligned}
$$

from which

$$
(12)\qquad |I_1| \leq \left|\frac{\sec\alpha}{\lambda z'^n}\right| \int^{t_1} t^{\lambda-2k+n}e^{-t}\left|1-\left(1-\frac{t}{t_1}\right)^{\lambda}\right| dt.
$$

Or if

$$
I_1' = \int_0^{t_1} t^{\lambda-2k+n}e^{-t}dt, \qquad I_1' < \Gamma(\lambda - 2k + n + 1),
$$

$$
(13)\qquad I_1'' = \int_0^{t_1} t^{\lambda-2k+n}e^{-t}\left(1-\frac{t}{t_1}\right)^{\lambda} dt,
$$

$$
(14)\qquad |I_1| \leq \left|\frac{\sec\alpha}{\lambda z'^n}\right| [I_1' + I_1''].
$$

To find an upper bound for I_1'', introduce the integrals

$$
J_1 = \int_0^{t_1/2} t^{\lambda-2k+n}e^{-t}(t_1-t)^{\lambda}dt,
$$

$$
J_2 = \int_{t_1/2}^{t_1} t^{\lambda-2k+n}e^{-t}(t_1-t)^{\lambda}dt.
$$

Then

$$
(15)\qquad I_1'' = t_1^{-\lambda}(J_1 + J_2).
$$

In J_1,

$$
t_1 - t \geq \frac{t_1}{2}, \qquad (t_1-t)^{\lambda} \leq \left(\frac{t_1}{2}\right)^{\lambda}.
$$

Consequently

$$
(16)\qquad J_1 < \left(\frac{t_1}{2}\right)^{\lambda}\Gamma(\lambda - 2k + n + 1).
$$

---

* In the following developments if the real power of a positive quantity is indicated, the absolute value will be understood.

Also

$$J_2 < e^{-(t_1/2)} t_1^{\lambda-2k+n} \cdot \int_{t_1/2}^{t_1} (t_1 - t)^\lambda \, dt < \frac{e^{-(t_1/2)} t_1^{2\lambda-2k+n+1}}{2^{\lambda+1}(\lambda+1)}.$$

Hence

$$(17) \quad |I_1| < \left| \frac{\sec\alpha}{\lambda z'^n} \right| \left\{ \Gamma(\lambda - 2k + n + 1)(1 + 2^{-\lambda}) + \frac{e^{-(t_1/2)} t_1^{\lambda-2k+n+1}}{2^{\lambda+1}(\lambda+1)} \right\}.$$

Since $-1 < -\lambda - 1$, we obtain from (9), (10),

$$(18) \quad |I_2| \leq \frac{\sec\alpha}{|z'|^n} \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} \left(1 - \frac{t}{t_1}\right)^\lambda \left[\int_0^{t/t_1} |(1 - \bar{x})^{-\lambda-1}| \, d\bar{x} \right] dt.$$

Also,

$$\int_0^{t/t_1} |(1 - \bar{x})^{-\lambda-1}| \, d\bar{x} = \int_0^1 (\cdots) d\bar{x} + \int_1^{t/t_1} (\cdots) d\bar{x}$$
$$= \frac{1}{|\lambda|} \left[1 + \left(\frac{t}{t_1} - 1 \right)^{-\lambda} \right].$$

Consequently from (18),

$$(19) \quad \begin{aligned} |I_2| &\leq \frac{\sec\alpha}{|z'|^n |\lambda|} \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} \left[1 + \left(\frac{t}{t_1} - 1 \right)^\lambda \right] dt \\ &\leq \frac{\sec\alpha}{|z'|^n |\lambda|} [I_2' + I_2''], \end{aligned}$$

$$(20) \quad I_2' = \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} dt < \Gamma(\lambda - 2k + n + 1),$$

$$(21) \quad I_2'' = t_1^{-\lambda} \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} (t - t_1)^\lambda dt.$$

Let $\bar{t} = t - t_1$.

$$(22) \quad \begin{aligned} I_2'' &= t_1^{-\lambda} \int_0^{\infty} (\bar{t} + t_1)^{\lambda-2k+n} e^{-\bar{t}-t_1} \bar{t}^\lambda d\bar{t} \\ &= t_1^{-\lambda} e^{-t_1} \int_0^{\infty} (t + t_1)^{\lambda-2k+n} e^{-t} t^\lambda dt. \end{aligned}$$

If $0 \leq t \leq t_1$, $t + t_1 \leq 2t_1$; if $t \geq t_1$, $t + t_1 \leq 2t$. Hence

$$\int_0^{t_1} (t + t_1)^{\lambda-2k+n} e^{-t} t^\lambda dt < (2t_1)^{\lambda-2k+n} \Gamma(\lambda + 1),$$

$$\int_{t_1}^{\infty} (t + t_1)^{\lambda-2k+n} e^{-t} t^\lambda dt < 2^{\lambda-2k+n} \Gamma(2\lambda - 2k + n + 1).$$

Consequently

$$I_2'' < t_1^{-\lambda}e^{-t_1}2^{\lambda-2k+n}[t_1^{\lambda-2k+n}\Gamma(\lambda+1)+\Gamma(2\lambda-2k+n+1)],$$

$$(23)\qquad |I_2| < \frac{\sec\alpha}{|z'|^n|\lambda|}\{\Gamma(\lambda-2k+n+1) + e^{-t_1}2^{\lambda-2k+n}t_1^{-\lambda}[\Gamma(\lambda+1)t_1^{\lambda-2k+n}+\Gamma(2\lambda-2k+n+1)]\}.$$

For large values of $|z'| = |z|$, and for $\sec\alpha < C$, the sum of the upper bounds for $|I_1|$ and $|I_2|$ is of the order of $|z|^{-n}$; since this sum is independent of y, for $\sec\alpha < C$, the asymptotic expansion (5) holds also in a sector ($\sec\alpha < C$) which includes the negative half of the real axis.*

2. Suppose given the equation† and inequalities

$$(24)\qquad \frac{d^2x}{dt^2} - \phi(t)x = 0,$$

$$\alpha^2t^{m_1} < \phi(t) < \alpha^2t^{m_2}, \qquad t > t_0, \quad m_1 > 0.$$

The auxiliary equation

$$(25)\qquad \frac{d^2y}{dt^2} - \alpha^2t^my = 0, \qquad m \neq -2$$

can be transformed into

$$(26)\qquad \frac{d^2\bar{y}}{dz^2} + \left[-\frac{1}{4}+\frac{\frac{1}{4}-p^2}{z^2}\right]\bar{y} = 0$$

by means of the substitutions

$$(27)\qquad p = \frac{1}{m+2}, \qquad y = t^{-(m/4)}\bar{y}, \qquad z = \frac{4\alpha}{m+2}t^{(m+2)/2}.$$

Consequently any solution of (25) can be represented in the form

$$y = t^{-(m/4)}\left[C_1W_{0,\,p}\left(\frac{4\alpha}{m+2}t^{(m+2)/2}\right) + C_2W_{0,\,p}\left(\frac{-4\alpha}{m+2}t^{(m+2)/2}\right)\right].$$

---

* From the preceding developments we obtain

$$W_{k,\,m}(z) = e^{-(z/2)}z^k\left\{a_1 + \frac{a_2}{z} + \cdots + \frac{a_{n+1}}{z^{n+1}} + R_{n+1}(z)\right\},$$

where $|R_{n+1}(z)| < A_{n+1}|z|^{-(n+1)}$. Consequently

$$W_{k,\,m}(z) = e^{-(z/2)}z^k\left\{a_1 + \cdots + \frac{a_n}{z^n} + \bar{R}_n(z)\right\},$$

$$|\bar{R}_n(z)| = \left|\frac{a_{n+1}}{z^{n+1}} + R_{n+1}(z)\right| \leq \frac{|a_{n+1}| + A_{n+1}}{|z^{n+1}|}.$$

Hence it can be assumed that $R_n(z)$ is of the order of $|z|^{-n-1}$.

† On certain linear differential equations of the second order, *Annals of Math.*, Vol. 24, No. 1, 1922.

Since $-1 < -\frac{1}{2} + p < 0$, the results of the first section can be applied. Also, from (1) it is easily shown that $W_{-0,\,m}(-z)$ is equal to a real function plus a solution of the order of $e^{-(z/2)}$ under the hypotheses of section 1. Hence the asymptotic expansion for $W_{0,\,p}(-z)$ is that of a real solution of (26).

Consequently by an argument exactly similar to that employed at the end of § 2 of the paper referred to, it is seen that if $Y_2(t)$ is the solution of (24) passing through (t_0, x_0) which remains bounded for $t < t_0$, Y_2', Y_2'', the corresponding solutions of (25) for $m = m_1$, $m = m_2$ respectively, then for $t > t_0$,

$$
\begin{gathered}
Y_2' < Y_2 < Y_2'', \\
Y_2^{(i)} = C_i t^{-(m_i/4)} e^{2\alpha/(m_i+2) t^{(m_i+2)/2}} \{1 + \overline{R}_1(t) t^{-(m_i+2)/2}\}.
\end{gathered}
\tag{28}
$$

Also, if $m_1 < m_2 < -2$, $p_i < 0$, and z approaches zero as $t \to \infty$. Inequalities (28) hold again, with

$$
\overline{Y}_2^{(i)} = C_i' t\{1 + \sum_{n=1}^{\infty} a_n t^{n(m_i+2)}\} + C_i''\{1 + \sum_{n=1}^{\infty} a_n' t^{n(m_i+2)/2}\},
$$

as is seen by expressing the solutions of (26) in terms of the functions $M_{0,\,p}(z)$, $M_{0,\,-p}(z)$.*

* Whittaker and Watson, l.c.

SOME GEOMETRIC APPLICATIONS OF SYMMETRIC SUBSTITUTION GROUPS.

BY ARNOLD EMCH.

I. INTRODUCTION.

The geometry of the symmetric group in its fundamental aspects has been investigated in an important memoir by J. Veronese.* Meanwhile the literature on invariant forms under finite collineation groups has become quite extensive.

The n letters $a_1, a_2, a_3, \cdots, a_n$ of a substitution may be considered as the homogeneous coördinates of a point of a projective $(n-1)$-space, so that to the $n!$ substitutions of the group correspond the same number of points which lie on the hyper-quadric

$$\sum_{i=1}^{n}(x_i^2)\cdot \sum_{\substack{i,k=1\\ i\neq k}}^{n}(a_ia_k) - \sum_{\substack{i,k=1\\ i\neq k}}^{n}(x_ix_k)\cdot \sum_{i=1}^{n}(a_i^2) = 0. \tag{1}$$

This may always be written in the form

$$\lambda\left(\sum_{i=1}^{n}x_i\right)^2 - \mu\sum_{\substack{i,k=1\\ i\neq k}}^{n}(x_i-x_k)^2 = 0. \tag{2}$$

For the purpose of this paper I shall quote in substance a few theorems for $n = 3$ and $n = 4$; i.e., for the groups G_6 and G_{24}.

THEOREM 1: *All sextuples of points of the G_6 lie on a pencil of conics, which touch the lines*

$$x_1 + \epsilon x_2 + \epsilon^2 x_3 = 0$$

and

$$x_1 + \epsilon^2 x_2 + \epsilon x_3 = 0$$

at their intersections I and J with the unit-line $e \equiv x_1 + x_2 + x_3 = 0$.

Denoting by $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$ the intersections of the sides $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ with e, by $A_1A_2A_3$ and E the coördinate-triangle and unit-point, we have

THEOREM 2: *The points of every sextuple determine* 3 *involutions on the corresponding conic, with $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$ as centers and $A_1E_1 = l_1$, $A_2E_2 = l_2$, $A_3E_3 = l_3$ respectively as axes of the involution. If these cut e in the same*

* "Interprétation géométrique de la théorie des substitutions de n letters, particulièrement pour $n = 3, 4, 5, 6$, en relation avec les groupes de l'Hexagramme mystique," *Annali di Matematica*, Vol. XI, Ser. II, pp. 93–236 (July, 1882).

order in $\mathfrak{E}_1'$, $\mathfrak{E}_2'$, $\mathfrak{E}_3'$, *then* $\mathfrak{E}_1\mathfrak{E}_1'$, $\mathfrak{E}_2\mathfrak{E}_2'$, $\mathfrak{E}_3\mathfrak{E}_3'$ *are pairs of an involution with* I *and* J *as double points.*

THEOREM 3: *There exists a group of collineations, simply isomorphic with the* G_6, *which leaves all conics of the pencil of the group invariant, and which permutes three associated involutions of a sextuple on every conic of the group.*

For the G_{24} we have

THEOREM 4: *Any set of* 24 *points of the* G_{24} *lies on* 16 *conics of a quadric whose planes by four pass through the four lines* s_i *cut out from the unit-plane* e *by the coördinate-planes* $x_i = 0$. *The points of the group lie two by two on* 72 *lines which in sets of* 12 *pass through the six vertices* $\mathfrak{E}_{ik}$ *of the quadrilateral* $s_1s_2s_3s_4$. 24 *points of the* S_{24} *form* 6 *involutions with the* $\mathfrak{E}_{ik}$*'s as centers and the* 6 *planes through the edges of the coördinate-tetrahedron and the unit-point, taken in the proper order, as axial planes. All quadrics of the* G_{24} *form a pencil and touch each other and the cone* $\Sigma(x_i - x_k)^2 = 0$ *along its intersection with* e.

$\mathfrak{E}_{ik}$ is the intersection of $\overline{A_iA_k}$ with e. These and related theorems may be immediately generalized for the $G_{n!}$, but nothing essentially new would be gained by doing so.

It is the purpose of this paper to study some of the curves and surfaces which are associated with these groups, i.e., the G_6 and G_{24}, and are invariant in the isomorphic groups of collineation.

Denoting a substitution of the collineation group G_6 by S_{ikl}, we have

$$S_{ikl} \equiv \begin{cases} \rho x_1' = x_i \\ \rho x_2' = x_k \\ \rho x_3' = x_l \end{cases} \quad \begin{array}{l} (i, k, l = 1, 2, 3) \\ (i \neq k \neq l). \end{array} \tag{3}$$

For the G_{24} we have

$$S_{ijkl} \equiv \begin{cases} \rho x_1' = x_i \\ \rho x_2' = x_j \\ \rho x_3' = x_k \\ \rho x_4' = x_l \end{cases} \quad \begin{array}{l} (i, j, k, l = 1, 2, 3, 4) \\ (i \neq j \neq k \neq l). \end{array} \tag{4}$$

II. INVARIANT PLANE n-ICS OF THE G_6.

§ 1. *General Case.*

1. Let $\varphi_1 = \Sigma x_i$, $\varphi_2 = \Sigma x_ix_k$, $\varphi_3 = x_1x_2x_3$ denote the elementary symmetric ternary forms, then every symmetric ternary n-ic, or curve of order n, of this type may be written in the form

$$\begin{aligned} C_n \equiv \lambda_0\varphi_1^n + \lambda_1\varphi_1^{n-2}\varphi_2 + \lambda_2\varphi_1^{n-3}\varphi_3 + \lambda_3\varphi_1^{n-4}\varphi_2^2 \\ + \lambda_4\varphi_1^{n-5}\varphi_2\varphi_3 + \lambda_5\varphi_1^{n-6}\varphi_2^3 + \lambda_6\varphi_1^{n-6}\varphi_3^2 + \cdots = 0. \end{aligned} \tag{5}$$

Such an n-ic is obviously invariant under the group of collineations G_6 represented by (3) and contains ∞^1 sextuples S_6 of points of the G_6.

Two n-ics of this kind intersect in n^2 points which group themselves into a finite number $\mu \leqq n/6$ of sextuples, while the rest of the common points, $n^2 - \mu$, are absorbed in a definite manner by $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, I, J as will be shown in the cases of cubics, quartics, quintics, and sextics.

The determination of the exact number of effective constants for the n-ic (5) is a well-known problem of partition in number theory.* If the exponents of φ_1, φ_2, φ_3 in the general term of (5) are α, β, γ, there is

$$\alpha + 2\beta + 3\gamma = n. \tag{6}$$

There are as many distinct terms in (5) as there are positive integral solutions α, β, γ, for a given n, of the diophantine equation (6).

If this number is N, the number of effective constants is $N - 1$. N is the largest positive integer which is equal or comes nearest to $(n + 3)^2/12$. As the six points of an S_6 determine three involutions on the conic K_6 associated with the S_6, with $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ as centers and $x_2 - x_3 = 0$, $x_3 - x_1 = 0$, $x_1 - x_2 = 0$ as axes of involution, to a point P on K_6 corresponds a point Q of S_6, so that $\overline{PQ}$ passes through $\mathfrak{C}_1$ and cuts $x_2 - x_3 = 0$ in a point R so that $(\mathfrak{C}_1 RPQ) = -1$. When P approaches a point R on the line $x_2 - x_3 = 0$, Q does the same thing. The same situation exists for the other involutions. Hence any conic K_6 of the S_6 cuts the lines $l_1 \equiv x_2 - x_3 = 0$, $l_2 \equiv x_3 - x_1 = 0$, $l_3 \equiv x_1 - x_2 = 0$ in six points so that the tangents to K_6 at these points, in pairs, pass through $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$. This property may be extended to any general invariant n-ic of the S_6, as will appear from the polar of $\mathfrak{C}_i$ later on, so that we have

THEOREM 5: *A general invariant n-ic C_n, $F(x_1, x_2, x_3) = 0$ cuts each of the three lines $x_i - x_k = 0$ in n points, so that the tangents to the n-ic at these points pass through the corresponding point $\mathfrak{C}_i$.*

If a point P_{123} lies on $x_1 - x_2 = 0$, the six points of the S_6 coincide by twos, i.e., $P_{123} = P_{213} = P_{113}$, $P_{132} = P_{231} = P_{131}$, $P_{312} = P_{321} = P_{311}$, and their joins pass, in the limit, through $\mathfrak{C}_3$, $\mathfrak{C}_2$, $\mathfrak{C}_1$. Hence the

THEOREM 6: *If an invariant conic K_6 is tangent to an invariant C_n at one of the points of intersection of the C_n with the lines l_1, l_2, l_3, say l_1, then K_6 touches the C_n in two other points which lie on l_2 and l_3. There are, in general, n such conics.*

That the C_n cuts, say l_1, in n points with the tangents at these points passing through $\mathfrak{C}_1$ is corroborated by the fact that the first polar of $\mathfrak{C}_1$ with respect to the C_n breaks up into the line l_1 and an $(n - 2)$-ic.

When a conic K_6 touches the C_n in a point which does not lie on a line

* See Dickson, "History of the Theory of Numbers," Vol. II, Chap. III, pp. 101–164.

l, then it touches the C_n in 5 other points. The six points of tangency of the K_6 and C_n form, of course, a sextuple of the S_6.

To determine the number of such conics it must be remembered that all K_6's of the S_6 form a pencil

$$\varphi_1^2 + \lambda\varphi_2 = 0.$$

Such a conic is tangent to the n-ic $F = 0$, when

$$2\varphi_1 + \lambda\frac{\partial\varphi_2}{\partial x_i} = \mu\frac{\partial F}{\partial x_i}, \qquad i = 1, 2, 3.$$

Now $\partial\varphi_2/\partial x_i = \varphi_1 - x_i$. Denoting $\partial F/\partial x_i$ by F_i, this leads to the condition

$$\begin{vmatrix} 2\varphi_1 & \varphi_1 - x_1 & F_1 \\ 2\varphi_1 & \varphi_1 - x_2 & F_2 \\ 2\varphi_1 & \varphi_1 - x_3 & F_3 \end{vmatrix} = 0,$$

or

$$x_1(F_2 - F_3) + x_2(F_3 - F_1) + x_3(F_1 - F_2) = 0.$$

Computing this by the use of (5), it reduces to the form

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)\Phi^{n-3}(x_1, x_2, x_3) = 0.$$

This is a degenerate n-ic which cuts the C_n in n^2 points which are points of tangency of conics of the group. Obviously, among these points are included the $3n$ points cut out by the C_n on the lines l. Outside of these points there are $n^2 - 3n$ points of tangency of conics with the C_n. In case that the C_n and the Φ^{n-3} have no points in common which lie on the unit-line, $n^2 - 3n$ is the number of tangencies of proper conics, and this number is a multiple of 6; there are then $\frac{1}{6}(n^2 - 3n)$ such conics touching the C_n in six points. If k points of intersection of F and Φ^{n-3} lie on the unit-line, the number of conics is reduced accordingly.

Summing up we may state the

THEOREM 7: *If k points of intersections of C_n and Φ^{n-3} are absorbed by points on the unit-line, then there are, in general, n invariant tri-tangent conics, with points of tangency on the lines $l_1l_2l_3$, and $\frac{1}{6}(n^2 - 3n - k)$ hexa-tangent invariant conics of the C_n.*

In case of a cubic there are merely 3 tritangent conics. For a sextic there are 6 tritangent conics and 3 hexatangent conics.

2. **Double Tangents and Double Points.**—The polar of $\mathfrak{E}_1$ with respect to the C_n is simply $F_2 - F_3 = 0$, or

$$(x_2 - x_3)\{\lambda_1\varphi_1^{n-2} + 2\lambda_3\varphi_1^{n-4}\varphi_2 + \lambda_4\varphi_1^{n-5}\varphi_3 + 3\lambda_5\varphi_1^{n-6}\varphi_2^2 + \cdots + x_1(\lambda_2\varphi_1^{n-3} + \lambda_4\varphi_1^{n-5}\varphi_2 + 2\lambda_6\varphi_1^{n-6}\varphi_3 + \cdots)\} = 0,$$

or

$$(x_2 - x_3)\psi_{n-2} = 0.$$

Hence, there are in general $n(n - 2)$ tangents from $\mathfrak{E}_1$ to the C_n whose

points of tangency P do not lie on the line $x_2 - x_3 = 0$. Suppose $\mathfrak{E}_1P$ as such a tangent. On account of the invariance of C_n, there is another point Q on C_n and l_1 which is a point of tangency.

If the C_n and ψ_{n-2} have intersections on the unit-line l, the number of tangents from the points $\mathfrak{E}_i$ is reduced by a certain number k. As every proper tangent from $\mathfrak{E}_i$ to the C_n touches C_n in another point, we may state the

THEOREM 8: *From every point $\mathfrak{E}_i$ there are $\frac{1}{2}\{n(n-2)-k\}$ double tangents to the C_n. There are altogether $\frac{3}{2}\{n(n-2)-k\}$ such double tangents.*

If an invariant n-ic has a multiple point at $\mathfrak{E}_1$, then it also has multiple points of the same type at $\mathfrak{E}_2$ and $\mathfrak{E}_3$. Likewise, multiplicities of the same sort appear simultaneously at A_1, A_2, A_3, and at I and J. According to theorem 5 a multiplicity in a point of the axes l_i of involutory perspective occurs when two branches come into contact at such a point. This may be an ordinary double point or a tacnode. If an n-ic passes through E it will have a singularity at E. Outside of these points the multiple points of an n-ic, if there are any, lie by groups of sextuples on conics of the G_6.

That E is a singularity for an invariant n-ic passing through E can easily be proved as follows: Suppose that b is a simple branch of such a curve through E and in its neighborhood whose tangent at E does not coincide with one of the l_i's. When a point P describes b the 5 equivalent points of P of the sextuple describe 5 other branches of the same curve through E. Hence E is a sextuple point of the n-ic. When b has one of the l_i's as a tangent at E, then as l_i is invariant in one collineation of the G_6, there will be a second branch of the curve having the same l_i as a tangent. The same situation exists for the other two l_i's. Hence the sextuple point may become a triple tacnode. Imposing the single condition that the curve shall have a double point and no higher singularity at E leads to the result that in such a case E is an isolated double point with $\overline{EI}$ and $\overline{EJ}$ as imaginary tangents. In fact this is verified by making C_n pass through E. This establishes a linear relation between all λ's in (5). When this obtains, $\partial C_n/\partial x_1$, $\partial C_n/\partial x_2$, $\partial C_n/\partial x_3$ vanish at E, which establishes the fact that E is a singularity. To ascertain its nature, we may take A_2EA_3 as the new coordinate-triangle, so that $x_1 = x_1'$, $x_2 = x_1' - x_2'$, $x_3 = x_1' - x_3'$. Substituting these in (5), the coefficients of $x_1'^n$, $x_1'^{n-1}$ vanish identically, and the coefficient of $x_1'^{n-2}$ contains $x_2'^2 - x_2'x_3' + x_3'^2$ as a factor. This breaks up into two linear factors which are the transformed equations of $\overline{EI}$ and $\overline{EJ}$, and which represent the tangents to C_n at E. If ϵ represents a cube root of unity the equations of $\overline{EI}$ and $\overline{EJ}$ turn out to be

$$\overline{EI} \equiv x_1 + \epsilon x_2 + \epsilon^2 x_3 = 0,$$
$$\overline{EJ} \equiv x_1 + \epsilon^2 x_2 + \epsilon x_3 = 0.$$

Summing up we have

THEOREM 9: *Singularities of invariant n-ics occur simultaneously at the A's, $\mathfrak{E}$'s, I and J, if they occur at all at any of these sets of points. Singularities on the l_i's occur in triples belonging to the G_6. If the n-ic passes through E, then, in general, E is an isolated double point of the n-ic. For curves higher than the sixth order, E may become a sextuple or multiple sextuple point, including triple tacnodal sets. Outside of these, singularities, if they exist, occur in sextuples lying on conics of the G_6.*

It is of importance to know whether there exist rational invariant n-ics. The question can be answered affirmatively as follows: We may restrict ourselves to the cases where the double points occur in groups of sextuples only. For this purpose there must be

$$\delta = \frac{(n-1)(n-2)}{2} = 6m,$$

where m is a positive integer. Choosing $n = 12k + 5$, there is

$$\delta = 6(3k+1)(4k+1)$$

and

$$m = (3k+1)(4k+1).$$

m denotes the number of sextuples of double points. Clearly the lowest order for an irreducible rational n-ic is 5. The six double points of the rational invariant quintic lie on a conic of the G_6. The general quintic belonging to the G_6 passes singly through $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, I, J.

That there are rational curves with other singularities than those in sextuples appears from the case of the sextic with double points at A_1, A_2, A_3, E and at the points of a given sextuple. Another case is the septimic with a sextuple point at E.

§ 2. *Cubic.*

The most general cubic of the S_6 may be written in the form

$$F = \varphi_1^3 + \lambda\varphi_1\varphi_2 + \mu\varphi_3 = 0. \tag{7}$$

It depends on two effective parameters λ, μ, and cuts the unit-line $e \equiv x_1 + x_2 + x_3 = 0$ in $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ which are points of inflexion of the cubic. The inflexional tangents at these points are

$$(\lambda + \mu)x_1 + \lambda x_2 + \lambda x_3 = 0,$$

$$\lambda x_1 + (\lambda + \mu)x_2 + \lambda x_3 = 0,$$

$$\lambda x_1 + \lambda x_2 + (\lambda + \mu)x_3 = 0.$$

If these are chosen as sides of a new coördinate triangle, F assumes the form

$$(x_1 + x_2 + x_3)^3 - \frac{(3\lambda + \mu)^3}{\lambda^3 + \lambda^2\mu - \mu^2} \cdot x_1x_2x_3 = 0. \tag{8}$$

Now it is known that every general (elliptic) cubic may be reduced to the form

$$x_1^3 + x_2^3 + x_3^3 - 3kx_1x_2x_3 = 0, \tag{9}$$

in which $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ are again inflexional points. If we choose the inflexional tangents again as sides of a new coördinate triangle, the general cubic assumes the form

$$(x_1 + x_2 + x_3)^3 - \frac{3(k + 2)^3}{k^2 + k + 1} \cdot x_1x_2x_3 = 0, \tag{10}$$

in which the parameter multiplying $x_1x_2x_3$ may have any value. But as, likewise, also the parameter (rational function of λ and μ) multiplying $x_1x_2x_3$ in the reduced form of the symmetric cubic may have any value, it is evident that the cubic invariant in the symmetric group S_6 is a general cubic.

There is however also a pencil of rational cubics in the G_6. If we make C_3 pass through E, by choosing $\mu = -27 - 9\lambda$, this pencil is

$$C_3 \equiv \varphi_1^3 - 27\varphi_3 + \lambda(\varphi_1\varphi_2 - 9\varphi_3) = 0. \tag{11}$$

§ 3. *Quartic.*

1. From § 1, 2, in case of a quartic,

$$\begin{aligned} C_4 &\equiv \lambda_0\varphi_1^4 + \lambda_1\varphi_1^2\varphi_2 + \lambda_2\varphi_1\varphi_3 + \lambda_3\varphi_2^2 = 0, \\ \psi_2 &= \lambda\varphi_1^2 + \lambda_2\varphi_1x_1 + 2\lambda_3\varphi_2 = 0. \end{aligned} \tag{12}$$

The two curves intersect in 8 points of which two are I and J on e ($k = 2$). Outside of e there are therefore 6 points of intersection, and consequently 3 double tangents from every point $\mathfrak{E}_i$. The unit line e is obviously a double tangent to the C_4 at I and J. In this manner we have accounted for 10 double tangents of the C_4.

Every $\mathfrak{E}_i$ is the center of an involutory perspective collineation, with $x_j - x_k = 0$ as the axis of perspective (j and k being the indices chosen from 1, 2, 3, different from i), by which the C_4 is transformed into itself.

The Hessian of C_4 which is an invariant C_6 cuts the C_4 in 24 points of inflexion which lie in sextuples on four conics of the G_6.

Choosing IJE as the new coördinate triangle, i.e., putting

$$\begin{aligned} x_1 + \epsilon x_2 + \epsilon^2 x_3 &= \rho x_1', \\ x_1 + \epsilon^2 x^2 + \epsilon x_3 &= \rho x_2', \\ x_1 + {}^{\cdot}x_2 + x_3 &= \rho x_3', \end{aligned}$$

the C_4 will assume the form

$$ax_3'^4 + bx_3'^2x_1'x_2' + cx_3'(x_1'^3 + x_2'^3) + dx_1'^2x_2'^2 = 0, \tag{13}$$

in which a, b, c, d are linear functions of $\lambda_0, \lambda_1, \lambda_2, \lambda_3$. This is precisely the form of the invariant quartic discussed by *Ciani.**

The triply infinite linear system of C_4's contains, of course also, the pencil

$$\sum x_i^4 + 6\lambda \sum x_i^2 x_k^2 = 0, \tag{14}$$

which is invariant under the 24 substitutions of the octahedral group

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ \pm x_i & \pm x_k & \pm x_j \end{pmatrix} \qquad i, k, j = 1, 2, 3, \quad i \neq k \neq j \neq i.$$

Among the curves of the pencil are two Kleinean curves, which are obtained for

$$\lambda = \frac{-1 + i\sqrt{7}}{4}, \qquad \lambda = \frac{-1 - i\sqrt{7}}{4}.$$

These cases are also discussed by Ciani, loc. cit.

2. **Double Tangents of Quartic.**—From the well-known fact that if α, β, ν, δ is a set of four of the 28 double tangents of a general quartic, whose points of contact lie on a conic φ, then the quartic has the form

$$\alpha\beta\nu\delta - \lambda\varphi^2 = 0, \tag{15}$$

it must be possible to put our general quartic in this form. Denoting by λ a parameter

$$C_4 = \lambda_0\varphi_1^4 + \lambda_1\varphi_1^2\varphi_2 + \lambda_2\varphi_1\varphi_3 + \lambda_3\varphi_2^2 = 0$$

may be written in the identical form

$$\varphi_1\{(\lambda_0\lambda^2 - \lambda_3)\varphi_1^3 + (\lambda_1\lambda^2 - 2\lambda_3\lambda)\varphi_1\varphi_2 + \lambda^2\lambda_2\varphi_3\} - \lambda_3(\varphi_1^2 + \lambda\varphi_2)^2 = 0.$$

Now $e \equiv \varphi_1$ is a double tangent, and from each $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ there are 3 double tangents. Sets of three out of these 9 may be chosen so that their product forms a symmetric cubic. Such a product is necessarily of the form

$$(\varphi_1 + \mu x_1)(\varphi_1 + \mu x_2)(\varphi_1 + \mu x_3) = (1 + \mu)\varphi_1^3 + \mu^2\varphi_1\varphi_2 + \mu^3\varphi_3.$$

Now it is possible to choose λ in such a manner that the cubic

$$(\lambda_0\lambda^2 - \lambda_3)\varphi_1^3 + (\lambda_1\lambda^2 - 2\lambda_3\lambda)\varphi_1\varphi_2 + \lambda^2\lambda_2\varphi_3 = 0$$

becomes reducible like the cubic with the parameter μ. For this purpose we must eliminate μ and ρ from the three equations

$$\lambda_0\lambda^2 - \lambda_3 = \rho(1 + \mu), \qquad \lambda_1\lambda^2 - 2\lambda_3\lambda = \rho\mu^2, \qquad \lambda_3\lambda^2 = \rho\mu^3.$$

* "1 varii tipi possibili di quartiche piane piu volte omologico-armoniche," *Rendiconti del Circolo Matematico di Palermo*, Vol. XIII, pp. 347–373 (1899).

This leads to the cubic in λ

$$[\lambda_0\lambda_2^2 - \lambda_1^2(\lambda_1 + \lambda_2)]\lambda^3 + [6\lambda_1^2\lambda_3 + 4\lambda_1\lambda_2\lambda_3]\lambda^2 - [\lambda_2^2\lambda_3 + 8\lambda_1\lambda_3^2 + 4(\lambda_1 + \lambda_2)\lambda_3^2]\lambda + 8\lambda_3^3 = 0.$$

There are therefore three triples of double tangents. Hence

THEOREM 10: *The* 9 *double tangents through the* $\mathfrak{E}_i$*'s form three triples, whose* 6 *points of contact, together with I and J of course, lie on* 3 *conics of the* G_6. *The remaining* 18 *double tangents form* 3 *sextuples each circumscribed to a conic of the* G_6. *The twelve points of contact of each sextuple lie on two conics of the group.*

3. **The Quartic as an Envelope of Cubics.**—The conic $\varphi_1^2 + \lambda\varphi_2 = 0$ cuts the cubic

$$(\lambda_0\lambda^2 - \lambda_3)\varphi_1^3 + (\lambda_1\lambda^2 - 2\lambda_3\lambda)\varphi_1\varphi_2 + \lambda_2\lambda^2\varphi_3 = 0$$

in 6 points which are points of tangency of the quartic and the cubic. For every value of λ there is such a sextuple, so that the C_4 may be generated as the envelope of the system of cubics

$$(\lambda_0\varphi_1^3 + \lambda_1\varphi_1\varphi_2 + \lambda_2\varphi_3)\lambda^2 - 2\lambda_3\varphi_1\varphi_2\lambda - \lambda_3\varphi_1^3 = 0. \tag{16}$$

In fact the discriminant of this cubic with respect to λ gives precisely the C_4. Through every point (x) there are evidently two cubics touching the C_4 along sextuples. When (x) is on the C_4, then the two cubics coincide.

THEOREM 11: *Every* C_4 *is enveloped by a system of invariant cubics of index* 2. *Through every sextuple S there are two cubics which touch the* C_4 *in points of sextuples. When S is on the* C_4 *the two tangent-cubics coincide.*

The other well-known systems of enveloping conics and cubics of the quartic do not belong to symmetric forms and shall therefore not be considered in this place.

§ 4. *Quintics.*

1. **System of Quintics and their Double Points.**—The general system of invariant quintics

$$\lambda_0\varphi_1^5 + \lambda_1^3\varphi_1^3\varphi_2 + \lambda_2\varphi_1^2\varphi_3 + \lambda_3\varphi_1\varphi_2^2 + \lambda_4\varphi_2\varphi_3 = 0 \tag{17}$$

depends on four effective constants. All quintics of the system pass through the five fixed points $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, I, J and have $\overline{EI}$ and $\overline{EJ}$ as common tangents at I and J. Two quintics intersect in 25 points, of which 7 are absorbed by the five fixed points. The remaining 18 intersections form three sextuples of the G_6. Four independent sextuples determine a quintic uniquely and the quintics through three fixed sextuples form a pencil.

As a double point absorbs three conditions, any point P in a general position may be taken as a double point of a quintic, so that also the five equivalent points of the group are double points.

THEOREM 12: *The points of a sextuple are therefore the double points of a rational quintic.*

There can be only one quintic with a given sextuple of double points, since two distinct quintics with these as common double points would intersect in 27 points, which is impossible.

The quintics of the set

$$\lambda_2\varphi_1^2\varphi_3 + \lambda_3\varphi_1\varphi_2^2 + \lambda_4\varphi_2\varphi_3 = 0 \tag{18}$$

have A_1, A_2, A_3 as double points. When $\lambda_4 = -3\lambda_2 - 9\lambda_3$, then also E becomes a double point (isolated).

2. **Quintics as Envelopes and Problems of Closure.**—Multiplying the quintic by φ_1 we obtain the reducible sextic

$$\lambda_0\varphi_1^6 + \lambda_1\varphi_1^4\varphi_2 + \lambda_2\varphi_1^3\varphi_3 + \lambda_3\varphi_1^2\varphi_2^2 + \lambda_4\varphi_1\varphi_2\varphi_3 = 0, \tag{19}$$

which by transformation

$$\rho y_1 = \varphi_1^3, \qquad \rho y_2 = \varphi_1\varphi_2, \qquad \rho y_3 = \varphi_3 \tag{20}$$

is mapped on the conic K

$$\lambda_0 y_1^2 + \lambda_1 y_1 y_2 + \lambda_2 y_1 y_3 + \lambda_3 y_2^2 + \lambda_4 y_2 y_3 = 0 \tag{21}$$

in the (y)-plane. To the line $\varphi_1 = 0$ in the (x)-plane corresponds the point $(0, 0, 1)$ on the conic. Obviously the points $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, I, J of the quintic are mapped into $(0, 0, 1)$.

Conversely to a line $\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0$ corresponds the cubic

$$\alpha_1\varphi_1^3 + \alpha_2\varphi_1\varphi_2 + \alpha_3\varphi_3 = 0. \tag{22}$$

To a conic in (y) corresponds a sextic in (x), and so forth. To a tangent t of (21) corresponds a cubic C_3 which touches the quintic along the points of a sextuple which absorb 12 of the 15 points of intersection of the cubic and the quintic. The remaining three points of intersection lie at $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$. As the conic (21) is enveloped by its system of tangents we have

THEOREM 13: *A given general quintic of G_6 is enveloped by a definite system of ∞^1 cubics belonging to G_6, so that through every sextuple there are, in general, two sixfold tangent cubics.*

The ∞^2 double tangent conics of K may be written in the form

$$(\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3)^2 + K = 0. \tag{23}$$

From the algebraic form of (23) it is easily seen that through two fixed points in (y) there are in general four double tangent conics to K. Hence, when we consider the ∞^1 double tangent conics (23) through a fixed point we obtain a system such that through every point there are four conics of the system.

Transforming back to the (x)-plane we have

THEOREM 14: *Every point in (y) determines a system of sextics which envelopes the quintic such that every sextic touches the quintic along the points of two sextuples. Through every sextuple in a general position there are four such tangent sextics.*

These sextics have the form

$$(\alpha_1\varphi_1^3 + \alpha_2\varphi_1\varphi_2 + \alpha_3\varphi_3)^2 + \varphi_1 C_5 = 0, \tag{24}$$

where C_5 denotes the quintic. In the intersection of the sextic and the quintic $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ count for two points each, so that the complete intersection consists of $2\cdot 2\cdot 6 + 2\cdot 3 = 30$ points.

It is evident that other projective properties of the conic have their equivalent on the quintic. For example, consider two conics K, say K_1 and K_2, with a Poncelet polygone of n sides inscribed in K_1 and circumscribed to K_2. Going back to the (x)-plane we have

THEOREM 15: *Given two quintics $C_5^{(1)}$, $C_5^{(2)}$ of the G_6. Through any sextuple S_0 of $C_5^{(1)}$ pass a cubic touching $C_5^{(2)}$ along the points of a sextuple and cutting $C_5^{(1)}$ in a second sextuple S_1. Through S_1 pass another cubic touching C_5^2 in the same manner and cutting C_5^1 in a third sextuple S_2; suppose that after continuing this process n times, S_n coincides with S_0. If this happens once, S_n will always coincide with S_0, no matter what sextuple S_0 we choose on C_5^1.*

§ 5. *Sextics.*

1. **Systems of Sextics and Double Points.**—A general sextic of the G_6:

$$\lambda_0\varphi_1^6 + \lambda_1\varphi_1^4\varphi_2 + \lambda_2\varphi_1^3\varphi_3 + \lambda_3\varphi_1^2\varphi_2^2 + \lambda_4\varphi_1\varphi_2\varphi_3 + \lambda_5\varphi_2^3 + \lambda_6\varphi_3^2 = 0, \tag{25}$$

depends on six effective constants, so that six independent sextuples determine a sextic completely. Five such sextuples determine a pencil. From this follows

THEOREM 16: *All sextics which pass through five independent fixed sextuples pass through a sixth fixed sextuple.*

The points of a sextuple are double points of all sextics of a definite system. In addition to such a sextuple of double points, a sextic may have double points at A_1, A_2, A_3 and E, so that the sextic becomes a rational sextic. There is just one sextic with these double points since two sextics with the same double points would intersect in 40 points. The sextic

$$\lambda_4\varphi_1\varphi_2\varphi_3 + \lambda_5\varphi_2^3 + \lambda_6\varphi_3^2 = 0 \tag{26}$$

has triple points at A_1, A_2, A_3. Moreover if we choose $\lambda_6 = -9\lambda_4 - 27\lambda_5$, we obtain a pencil of rational sextics

$$\varphi_1\varphi_2\varphi_3 - 9\varphi_3^2 - \lambda(27\varphi_3^2 - \varphi_2^3) = 0, \tag{27}$$

which at A_1, A_2, A_3 have the common tangents $x_2 + x_3 = 0$, $x_3 + x_1 = 0$, $x_1 + x_2 = 0$, and at E the common tangents $\overline{EI}$ and $\overline{EJ}$.* Of the 36 points of intersection of two sextics 10 are absorbed by each A_1, A_2, A_3 (9 on account of the triple point, 1 on account of the common tangents at each point); 6 by E (double points with common tangents).

2. **Sextics as Envelopes.**—To the reducible nonic consisting of the product of φ_1^3 and the sextic corresponds in the (y)-plane, by the transformation (20), the cubic

$$\lambda_0 y_1^3 + \lambda_1 y_1^2 y_2 + \lambda_2 y_1^2 y_3 + \lambda_3 y_1 y_2^2 + \lambda_4 y_1 y_2 y_3 + \lambda_5 y_2^3 + \lambda_6 y_1 y_3^2 = 0, \tag{28}$$

which, in general, is elliptic. To the line $\varphi_1 = 0$ corresponds the point $(y) = (0, 0, 1)$, to the factor φ_1^3 of the nonic this point three times. To the intersections of the sextic with $\varphi_1 = 0$ corresponds this same point. To every sextuple of the sextic corresponds a point of the cubic (28). Conversely to every point of the cubic corresponds a sextuple of the sextic. More generally, to every sextuple in (x) corresponds a point in (y), and conversely. To lines, conics, etc., in (y) correspond symmetric cubics, sextics, etc., in (x). Hence, to the geometry of points, lines, conics, . . . in (y) corresponds abstractly the same geometry of sextuples, cubics, sextics, . . . of the G_6 in (x).

By means of this correspondence we are able to state immediately a number of theorems in the (x)-plane which are the equivalents of those in the (y)-plane. We shall restrict ourselves to some of the most important.

An elliptic cubic has 9 inflexions which lie 3 by 3 on 12 lines. In the (x)-plane we have

THEOREM 17: *There are* 9 *cubics which osculate a given (general) sextic of the* G_6 *in points of a sextuple. The nine sextuples of osculating points lie* 3 *by* 3 *on* 12 *cubics.*

Again an elliptic cubic admits of 27 conics with sextactic contact; hence

THEOREM 18: *There are* 27 *sextics which touch a given sextic in sextuples of sextactic points.*

With every inflexion of an elliptic cubic is associated a system of ∞^2 tritangent conics. Through every point in a general position there is a system of ∞^1 such conics, which envelope the cubic. In the (x)-plane we have accordingly

THEOREM 19: *A given sextic may be generated in nine ways by* ∞^2 *systems of enveloping sextics. Every enveloping sextic touches the given sextic in the points of three sextuples.*

* For other special types of sextics invariant under the G_6, for example the G_{360}, see A. B. Coble, "An Invariant Condition for Certain Automorphic Algebraic Forms." AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVIII, pp. 333–366 (1906).

There are 9 systems of ∞^1 doubly osculating conics for a given cubic. Thus

THEOREM 20: *Every sextic admits of 9 systems of ∞^1 doubly osculating sextics. Every enveloping sextic osculates the given sextic in the points of two sextuples.*

3. **Problems of Closure.**—Let P and Q be two points on the cubic C_3. Through P draw any line l cutting C_3 in A_l and B_l. Let the join B_lQ cut C_3 in a third point C_l; let C_lP cut C_3 in D_l; finally, let D_lQ cut C_3 in E_l. If E_l coincides with A_l, then this coincidence will take place, no matter what initial line l we draw through P. We have accordingly

THEOREM 21: *Let P and Q be two sextuples on the sextic C_6. Through P draw any cubic l cutting C_6 in the sextuples A_l and B_l. Let the cubic through B_l and Q cut C_6 in a third sextuple C_l; let the cubic through C_l and P cut C_6 in the sextuple D_l; finally let the cubic through D_l and Q cut C_6 in the sextuple E_l. If E_l coincides with A_l, then this coincidence will take place, no matter what initial cubic l we pass through P.*

Other equivalent theorems might be stated with equal ease.

III. INVARIANT SURFACES AND CURVES OF THE G_{24}.

§ 1. *The General Invariant n-ic.*

Denoting the elementary symmetric functions in the quaternary field again by

$$\varphi_1 = \Sigma x_i, \qquad \varphi_2 = \Sigma x_i x_k, \qquad \varphi_3 = \Sigma x_i x_j x_k, \qquad \varphi_4 = x_1 x_2 x_3 x_4,$$

the general n-ic may be written in the form

$$\sum_{i=0}^{N-1} \lambda_i \varphi_1^{\alpha} \varphi_2^{\beta} \varphi_3^{\gamma} \varphi_4^{\delta} = 0, \tag{29}$$

$$\alpha + 2\beta + 3\gamma + 4\delta = n, \tag{30}$$

so that the number N of effective constants is equal to the positive integral solution of this diophantine equation, diminished by one. It is not difficult to find the number N for a given numerical integral value of n. For example, the systems of quadrics, cubics, quartics, quintics, sextics depend on 1, 2, 4, 5, 8 effective constants.

In space of $m - 1$ dimensions the symmetric n-ic ($n \geqq m$)

$$\sum_{i=0}^{N-1} \varphi_1^{\alpha} \varphi_2^{\beta} \varphi_3^{\nu} \cdots \varphi_m^{\mu} = 0 \tag{31}$$

depends on N effective constants, whose number depends analogously on the partition problem in number theory.

$$\alpha + 2\beta + 3\nu + \cdots + m\mu = n. \tag{32}$$

In what follows I shall restrict myself to a short discussion of cubics and the sextic curves obtained as intersections of quadrics and cubics of the G_{24}.

§ 2. *The 27 Lines on a Symmetric Cubic.*

The symmetric cubic C_3

$$\varphi_1^3 + \lambda\varphi_1\varphi_2 + \mu\varphi_3 = 0, \tag{33}$$

or

$$\begin{aligned} &(x_1 + x_2 + x_3 + x_4)^3 \\ &+ \lambda(x_1 + x_2 + x_3 + x_4)[(x_1 + x_2)(x_3 + x_4) + x_1x_2 + x_3x_4] \\ &\qquad + \mu[(x_1 + x_2)x_3x_4 + (x_3 + x_4)x_1x_2] = 0 \end{aligned} \tag{34}$$

is satisfied by any point of the three lines

$$l_1\begin{cases} x_1 + x_2 = 0 \\ x_3 + x_4 = 0 \end{cases} \qquad l_2\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_4 = 0 \end{cases} \qquad l_3\begin{cases} x_1 + x_4 = 0 \\ x_2 + x_3 = 0. \end{cases}$$

These lines lie on the unit-plane $x_1 + x_2 + x_3 + x_4 = 0$ and form one of the 45 triangles of the cubic. To find the remaining 24 lines, pass any plane $x_3 + x_4 = \theta(x_1 + x_2)$ through l_1. This will cut C_3 in a conic whose projection upon the $(x_1x_2x_3)$-plane is obtained by the elimination of x_4. There is $x_4 = \theta(x_1 + x_2) - x_3$, so that (34) becomes, after dividing through by $(x_1 + x_2)$ and rearranging,

$$\begin{aligned} &[(1 + \theta)^3 + \lambda(1 + \theta)\theta]x_1^2 + [(1 + \theta)^3 + \lambda(1 + \theta)\theta]x_2^2 - [\lambda(1 + \theta) + \mu]x_3^2 \\ &\quad + 2\left[(1 + \theta)^3 + \lambda(1 + \theta)\theta + \frac{\lambda}{2}(1 + \theta) + \frac{\mu}{2}\theta\right]x_1x_2 \\ &\quad + 2\left[\frac{\lambda}{2}(1 + \theta)\theta + \frac{\mu}{2}\theta\right]x_1x_3 + 2\left[\frac{\lambda}{2}(1 + \theta)\theta + \frac{\mu}{2}\theta\right]x_2x_3 = 0. \end{aligned}$$

This conic degenerates into two lines, when the discriminant

$$\begin{vmatrix} [(1+\theta)^3+\lambda(1+\theta)\theta] & \left[(1+\theta)^3+\lambda(1+\theta)\theta+\frac{\lambda}{2}(1+\theta)+\frac{\mu}{2}\theta\right] & \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right] \\ \left[(1+\theta)^3+\lambda(1+\theta)\theta+\frac{\lambda}{2}(1+\theta)+\frac{\mu}{2}\theta\right] & [(1+\theta)^3+\lambda(1+\theta)\theta] & \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right] \\ \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right] & \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right] & -[\lambda(1 + \theta) + \mu] \end{vmatrix} = 0.$$

Subtracting the second from the first line and factoring we get

$$\begin{matrix} \frac{1}{2}[\lambda(1 + \theta) + \mu\theta]\cdot \\ [\lambda(1 + \theta) + \mu]\cdot \end{matrix} \begin{vmatrix} -1 & 1 & 0 \\ A & B & C \\ \frac{\theta}{2} & \frac{\theta}{2} & -1 \end{vmatrix} = 0$$

or

$$(1+\theta)[\lambda(1+\theta)+\mu\theta][\lambda(1+\theta)+\mu][4(1+\theta)^2+4\lambda\theta+\lambda+\mu\theta+\lambda\theta^2]=0.$$

The last factor may be written

$$(4+\lambda)\theta^2+(8+4\lambda+\mu)\theta+4+\lambda.$$

This equated to zero gives for the roots

$$\theta_{4,\,5}=\frac{-8-4\lambda-\mu\pm\sqrt{(8+4\lambda+\mu)^2-4(4+\lambda)^2}}{8+2\lambda}.$$

There are, therefore, 5 planes through l_1 which cut C_3 in pairs of lines. The parameters of these planes are

$$\theta_1=-1,\qquad \theta_2=-\frac{\lambda}{\lambda+\mu},\qquad \theta_3=-\frac{\lambda+\mu}{\lambda},$$

$$\theta_4=\frac{-8-4\lambda-\mu+\sqrt{(8+4\lambda+\mu)^2-(8+2\lambda)^2}}{8+2\lambda},$$

$$\theta_5=\frac{8+2\lambda}{-8-4\lambda-\mu+\sqrt{(8+4\lambda+\mu)^2-(8+2\lambda)^2}}.$$

The plane $\theta_1=-1$, of course, cuts C_3 in l_2 and l_3. As θ_2 and θ_3, as well as θ_4 and θ_5, are reciprocal to each other, the planes θ_2 and θ_3 are permuted by the substitutions $\binom{1234}{3412}$; $\binom{1234}{4312}$; $\binom{1234}{4321}$; $\binom{1234}{3412}$. The same substitutions permute θ_4 and θ_5. The same equivalent collineations transform θ_2 into θ_3, and θ_4 into θ_5. On the other hand, the substitutions (collineations)

$$\binom{1234}{1243}\quad\binom{1234}{2134}\quad\binom{1234}{2143}$$

leave each of those planes invariant.

On account of the symmetric character of the equation of C_3, precisely the same parameters and substitutions for the planes through l_2 and l_3, cutting C_3 in pairs of lines, are obtained. Hence

THEOREM 22: *The effective determination of the* 27 *lines of a symmetric cubic is possible by the solution of three linear and one quadratic equation with a parameter θ as the unknown.*

The three lines l_1, l_2, l_3 are left invariant or are permuted by the 24 substitutions of the symmetric group. Every substitution which permutes l_i and l_k $(i, k = 1, 2, 3)$ also permutes the planes through l_i and l_k with the same parameters.

§ 3. *Sextic Curves of the G_{24}.*

Two cubics

(35) $$F=a\varphi_1^3+b\varphi_1\varphi_2+c\varphi_3=0,$$

(36) $$G=d\varphi_1^3+e\varphi_1\varphi_2+f\varphi_3=0$$

of the set of symmetric quaternary cubics intersect in a space curve of order 9 which degenerates into a sextic S and three fixed lines C_3 ($\varphi_1 = 0$, $\varphi_3 = 0$) in the unit-plane.

$$Q^* = \frac{fF - cG}{\varphi_1} = (af - cd)\varphi_1^2 - (ce - bf)\varphi_2 = 0 \tag{37}$$

is a quadric through S. Conversely when a definite quadric

$$Q = \varphi_2 - \lambda\varphi_1^2 = 0 \tag{38}$$

is given, any cubic through the sextic on Q and F may be written in the form

$$(a + \lambda b)\varphi_1^3 + c\varphi_3 = 0. \tag{39}$$

From this follows that all possible sextics of the group on the quadric Q are cut out by a pencil of cubics which osculate along the three lines C_3.

On every sextic there is a simply infinite set of 24 points of the group which lie two by two on 12 lines through each of the $\mathfrak{E}_{ik}$'s. Hence, from each of these points the sextic is projected upon a plane into a curve all of whose points are double points, hence into a cubic. From this follows

THEOREM 23: *The sextics of the group are of genus* 4 *and form a set. Every sextic of the group lies on* 6 *cubic cones with their vertices at the* $\mathfrak{E}_{ik}$'s.

It is not difficult to find the equations of these cones. For example if we write $Q = \varphi_1^2 + \lambda\varphi_2$, $F = \varphi_1^3 + \mu\varphi_1\varphi_2 + \nu\varphi_3$, the cone with $\mathfrak{E}_{34}$ as a vertex has the form

$$\lambda(\varphi_1^3 + \mu\varphi_1\varphi_2 + \nu\varphi_3) - (\varphi_1^2 + \lambda\varphi_2)\{\mu(x_1 + x_2) + (\mu + \nu)(x_3 + x_4)\} = 0. \tag{40}$$

The relation between the six cones may be stated in

THEOREM 24: *The cubic cone through the sextic, with* $\mathfrak{E}_{ij}$ *as a vertex, osculates the plane* $x_i + x_j - x_k - x_l = 0$ *along the line* $\mathfrak{E}_{ij}\mathfrak{E}_{kl}$. *Two cubic cones through the sextic with* $\mathfrak{E}_{ij}$ *and* $\mathfrak{E}_{ik}$ *as vertices intersect moreover in a plane cubic which lies in the plane* $x_j - x_k = 0$.

UNIVERSITY OF ILLINOIS.

ON ELLIPTIC CYLINDER FUNCTIONS OF THE SECOND KIND.

By Sasindrachandra Dhar.

1. The canonical form of Mathieu's differential equation is given by

$$\frac{d^2y}{dz^2} + (A + 16q \cos 2z)y = 0. \tag{1}$$

For certain values of "A." two kinds of solutions of the above differential equation have been constructed. The periodic solutions of the first kind have been denoted by Professor Whittaker* in the forms:

$$\left.\begin{array}{l} ce_0(z, q),\ ce_1(z, q),\ \cdots\ ce_m(z, q),\ \cdots \\ \qquad\qquad se_1(z, q),\ \cdots\ se_m(z, q),\ \cdots \end{array}\right\}. \tag{2}$$

The solutions of the second kind corresponding to the above solutions of the first kind were first systematically studied by Mr. E. Lindsay Ince,† who gave us two methods for constructing the series of integrals. These like the solutions of the first kind are not, however, periodic. The special value of "A" for which the solution $ce_m(z, q)$ has been constructed is given by

$$A = m^2 + \frac{32q^2}{m^2 - 1} - \frac{128(5m^2 + 7)q^4}{(m^2 - 1)^3(m^2 - 4)} - \cdots, \text{ etc.} \tag{3}$$

We shall denote it, however, as

$$A = a_0 + a_1 q + a_2 q^2 + \cdots, \text{ etc.,} \tag{4}$$

where

$$a_0 = m^2, \qquad a_1 = 0, \qquad a_2 = \frac{32}{m^2 - 1}, \qquad \cdots, \qquad \text{etc.}$$

2. The existence of an infinite number of solutions of the second kind corresponding to the infinite number of solutions of the first kind can be easily demonstrated by the following well-known theorem of linear differential equation of the second order:

If $y = v$ be a particular integral of the differential equation

$$\frac{d^2y}{dz^2} + Qy = 0,$$

then the most general solution of the above equation is given by

$$y = v\left(B + C\int \frac{1}{v^2}\,dz\right).$$

* Whittaker, *Fifth International Congress of Mathematics*, 1912.

† E. Lindsay Ince, *Proc. Edin. Math. Soc.*, Vol. XXXIII, 1914–15.

Therefore the solution of the second kind corresponding to the solution $y = v$, of the first kind, is given by

$$y = Cv \int \frac{1}{v^2} dz, \tag{5}$$

where C is an arbitrary constant.

3. Mr. Ince has by using the above formula (5) calculated some of the simplest of the integrals. He has, in fact, calculated the integral corresponding to $ce_0(z, q)$. But the process is very laborious in practice and even then one cannot get as many terms as one would like.

His other method is comparatively easy, but requires for the determination of the solutions a knowledge of the forms of the integrals, which will be furnished by the formula (5). But all these integrals can be easily constructed by proceeding with a little modification, the method employed by Mathieu* and Sieger.† We proceed thus:

If, for instance, we require to construct the integral of the second kind corresponding to $ce_m(z, q)$, we will take the expression for "A" to be that given in (4) and assume

$$y = f_0(z) + qf_1(z) + q^2f_2(z) + \cdots \text{ etc.}, \tag{6}$$

where $f_0(z), f_1(z), f_2(z), \cdots$, etc., are functions of z only.

Substituting these values of A and y in the differential equation (1) and equating the different powers of q to zero, we get the following differential equations from which to determine $f_0(z), f_1(z), f_2(z), \cdots$, etc.:

$$(7)\quad \left\{\begin{array}{l} f_0''(z) + a_0f_0(z) = 0, \\ f_1''(z) + a_0f_1(z) = -\ \{a_1f_0(z) + 16 \cos 2zf_0(z)\}, \\ f_2''(z) + a_0f_2(z) = -\ \{a_2f_0(z) + a_1f_1(z) + 16 \cos 2zf_0(z)\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_n''(z) + a_0f_n(z) = -\ \{a_nf_0(z) + a_{n-1}f_1(z) \\ \qquad\qquad + \cdots + a_1f_{n-1}(z) + 16 \cos 2zf_{n-1}(z)\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right.$$

4. If, now, we solve the differential equation

$$f''(z) + a_0f_0(z) = 0,$$

we find that there are two possible values for $f_0(z)$, viz., cos mz and sin mz. If we proceed with $f_0(z) = \cos mz$, it will only enable us to obtain the solution $ce_m(z, q)$, for it is in this way that $ce_m(z, q)$ can be obtained, the constants $a_1, a_2, \cdots$, etc., being determined by the fact that $ce_m(z, q)$ is to be periodic.

* E. L. Mathieu, *Liouville's Journal* (2), XIII (1868).

† Sieger, *Annalen der Physik*, Bd. 27.

We will, therefore, proceed with $f_0(z) = \sin mz$, and solve the equations one by one.

In determining $f_1(z), f_2(z), \cdots$, etc., from the differential equations given in (7), it will be found necessary to put the expressions on the right-hand side of those equations in series of cosines or sines of multiples of z and further it will be seen that we shall be constantly required to find particular integrals of equations of the types:

$$\left.\begin{array}{ll} \text{(i) } y'' + m^2y = C\cos(m+\alpha)z, & \text{(ii) } y'' + m^2y = C\sin(m+\alpha)z, \\ \text{(iii) } y'' + m^2y = C\cos mz, & \text{(iv) } y'' + m^2y = C\sin mz, \\ \text{(v) } y'' + m^2y = Cz\cos\alpha z, & \text{(vi) } y'' + m^2y = Cz\sin\alpha z, \end{array}\right\} \quad (8)$$

whose particular integrals are given by

$$\left.\begin{array}{l} \text{(i) } y = -C\cos(m+\alpha)z/\alpha(2m+\alpha), \\ \text{(ii) } y = -C\sin(m+\alpha)z/\alpha(2m+\alpha), \\ \text{(iii) } y = Cz\sin mz/2m, \\ \text{(iv) } y = -Cz\cos mz/2m, \\ \text{(v) } y = \dfrac{C}{m^2-\alpha^2}z\cos\alpha z + \dfrac{2C\alpha}{(m^2-\alpha^2)^2}\sin\alpha z, \\ \text{(vi) } y = \dfrac{C}{m^2-\alpha^2}z\sin\alpha z - \dfrac{2C\alpha}{(m^2-\alpha^2)^2}\cos\alpha z. \end{array}\right\} \quad (9)$$

Thus, following the above processes, all the integrals of the second kind corresponding to those of the first kind as given in (2) can be obtained very easily. Following the notation suggested by Professor Whittaker, they may be denoted as

$$\left.\begin{array}{r} i\eta_0(z, q),\ i\eta_1(z, q),\ i\eta_2(z, q),\ \cdots\ i\eta_m(z, q),\ \cdots \\ j\eta_1(z, q),\ j\eta_2(z, q),\ \cdots\ j\eta_m(z, q),\ \cdots. \end{array}\right\} \quad (10)$$

5. Let us illustrate the processes indicated above by working out a particular case. Suppose we wish to find the integral which corresponds to $ce_1(z, q)$. The particular value of "A" for this is given by

$$A = 1 - 8q - 8q^2 + 8q^3 - \frac{8}{3}q^4 - \frac{88}{9}q^5 + \cdots, \text{ etc.} \quad (11)$$

Hence to find $i\eta_1(z, q)$, we shall have to solve the equations:

$$\left.\begin{array}{l} \text{(i) } f_1''(z) + f_1(z) = 8\sin z - 16\cos 2z\cdot\sin z, \\ \text{(ii) } f_2''(z) + f_2(z) = 8\sin z + 8f_1(z) - 16\cos 2z\cdot f_1(z), \\ \text{(iii) } f_3''(z) + f_3(z) = -8\sin z + 8f_1(z) + 8f_2(z) - 16\cos 2z\cdot f_2(z), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots. \end{array}\right\} \quad (12)$$

(*a*) Now, the solution of the equation (i) is obtained by adding up the particular integrals of

$$y''(z) + y(z) = 16\sin z; \qquad y''(z) + y(z) = -8\sin 3z,$$

which by the help of (8) and (9) is given by

$$f_1(z) = -\,8z\cos z + \sin 3z.$$

(*b*) To find $f_2(z)$, we need only find the particular integrals of the following equations:

$$y''(z) + y(z) = 64z\cos 3z; \qquad y''(z) + y(z) = 8\sin 3z;$$

and

$$y''(z) + y(z) = -\,8\sin 5z,$$

and add them up. Thus we get

$$f_2(z) = 8z\cos 3z + 5\sin 3z + \frac{1}{3}\sin 5z.$$

(*c*) To find $f_3(z)$, we transform the expression on the right-hand side of (iii) (12) as

$$-\,48\sin z + \frac{136}{3}\sin 3z - \frac{112}{3}\sin 5z - \frac{8}{3}\sin 7z - 16z\cos 3z - 64z\cos 5z,$$

and the form of $f_3(z)$ is determined by adding up the particular integrals of

$$y''(z) + y(z) = -\,48\sin z,\ \cdots,\ \text{etc.}, \qquad y''(z) + y(z) = -\,64z\cos 5z,$$

and hence we have

$$f_3(z) = 24z\cos z + 8z\cos 3z - \frac{8}{3}z\cos 5z + \frac{1}{18}\sin 7z + \frac{8}{3}\sin 5z - \frac{35}{3}\sin 3z.$$

Proceeding thus, we can get as many terms as we like and hence on arranging, we find $i\eta_1(z, q)$ to be given by

$$-\,8q(1 - 3q^2 + \cdots)z\left\{\cos z + q\cos 3z + q^2\left(-\cos 3z + \frac{1}{3}\cos 5z\right) + \cdots\right\}$$

$$+ \sin z + q\sin 3z + q^2\left(\frac{1}{3}\sin 5z + \sin 3z\right)$$

$$+ q^3\left(\frac{1}{18}\sin 7z + \frac{8}{3}\sin 5z - \frac{35}{3}\sin 3z\right) + \cdots,\ \text{etc.}$$

ON A NEW METHOD OF CONSTRUCTING SOLUTIONS OF THE SECOND KIND.

6. The methods given above and as also employed by Mr. E. Lindsay Ince are not suitable for studying the convergence of the series; but what is given below, while allowing us to construct the series of integrals very easily, is also suitable for the consideration of their convergency. This latter method follows lines similar to that employed by Frobenius* in solving linear differential equations and also similar to that employed by Professors Whittaker and Watson† for constructing integrals of the first kind.

* Frobenius, *Crelle's Journal*, Vol. LXXVI.

† Whittaker and Watson, "Modern Analysis," pp. 413–415.

Let us investigate the solution $i\eta_m(z, q)$ corresponding to the solution $ce_m(z, q)$ of the first kind, for which the value of "A" is given by (4).

If, now, we put in Matthieu's differential equation (1)

$$A = m^2 + 8p,$$

it will become

$$\frac{d^2y}{dz^2} + m^2y = -8(p + 2q\cos 2z)y.$$

When "p" and "q" are neglected, solutions of the equation are given by

$$y = \cos mz \qquad \text{and} \qquad y = \sin mz.$$

If we proceed with $y = \cos mz$, it will enable us to construct the solution $ce_m(z, q)$* and so we proceed with $y = \sin mz$. Let us denote $U_0(z) = \sin mz$. Then to obtain a closer approximation, we write $-8(p + 2q\cos 2z)U_0(z)$ as a series of sines of multiples of "z" in the form

$$-8\{q\sin(m-2)z + p\sin mz + q\sin(m+2)z\}$$

which we will denote by $V_1(z)$.

Then instead of solving the differential equation

$$\frac{d^2y}{dz^2} + m^2y = V_1(z),$$

we will solve the equation

$$\frac{d^2y}{dz^2} + m^2y = W_1(z), \tag{13}$$

where $W_1(z) = V_1(z) + (8p - a_1q)\sin mz$. Its integral, which we will denote by $U_1(z)$, is given by

$$U_1(z) = \frac{-2q\sin(m-2)z}{m-1} + \frac{2q\sin(m+2)z}{m+1} + \frac{a_1qz\cos mz}{2m}. \tag{14}$$

To obtain a still closer approximation, we will express $-8(p + 2q\cos 2z)U_1(z)$ as a series of sines of multiples of "z," which we will denote by $V_2(z)$, viz.,

$$\begin{aligned} V_2(z) = {} & \frac{16q^2\sin(m-4)z}{m-1} + \frac{16pq\sin(m-2)z}{m-1} + \frac{32q^2\sin mz}{m^2-1} \\ & - \frac{16pq\sin(m+2)z}{m+1} - \frac{16q^2\sin(m+4)z}{m+1} - \frac{8q^2a_1z\cos(m-2)z}{2m} \\ & - \frac{8pqa_1z\cos mz}{2m} - \frac{8q^2a_1z\cos(m+2)z}{2m}. \end{aligned} \tag{15}$$

Here again we solve the equation

$$\frac{d^2y}{dz^2} + m^2y = W_2(z),$$

* "Modern Analysis," p. 413.

where $W_2(z) = V_2(z) - a_2q^2 \sin mz + \lambda_2 z \cos mz$, where λ_2 has been determined in such a way that $W_2(z)$ does not involve "$z \cos mz$." *

Suppose $U_2(z)$ is the integral of the above equation. We will now proceed exactly with $U_2(z)$ as we have done with $U_1(z)$ and obtain the integral $U_3(z)$.

7. Continuing thus, we get the integrals $U_0(z)$, $U_1(z)$, $U_2(z)$, $\cdots$ $U_n(z)$, $\cdots$, etc., of the differential equations

$$\left.\begin{array}{l} \dfrac{dz^2}{d^2y} + m^2y = 0, \\ \dfrac{d^2y}{dz^2} + m^2y = W_1(z), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{d^2y}{dz^2} + m^2y = W_n(z), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \tag{16}$$

respectively, where

$$W_n(z) = V_n(z) - a_nq^n \sin mz + \lambda_n z \cos mz; \quad (m > 2)$$
$$V_n(z) = -8(p + 2q \cos 2z)U_{n-1}(z), \quad (n \geq 1).$$

Therefore, from (16), we have

$$\left(\frac{d^2}{dz^2} + m^2\right) \cdot \sum_{n=0}^{\infty} U_n(z) = \sum_{n=1}^{\infty} W_n(z),$$

$$\text{i.e.} = \sum_{n=1}^{\infty} V_n(z) + (8p - \sum_{1}^{\infty} a_nq^n) \sin mz + \sum_{n=2}^{\infty} \lambda_n z \cos mz,$$

or

$$\left\{\frac{d^2}{dz^2} + (A + 16q \cos 2z)\right\} \cdot \sum_{0}^{\infty} U_n(z)$$
$$= (8p - \sum_{1}^{\infty} a_nq^n) \sin mz + \sum_{2}^{\infty} \lambda_n z \cos mz.$$

But we have from (4)

$$8p = \sum_{1}^{\infty} a_nq^n,$$

and it will also be found that $\sum_2^\infty \lambda_n$ vanishes for the above value of "p." Hence if the series $\sum_0^\infty U_n(z)$ be uniformly convergent, the series will be a solution of Mathieu's equation. It is, in fact, the solution of the second kind, corresponding to $ce_m(z, q)$, as has been obtained by Mr. Lindsay Ince, that is,

$$i\eta_m(z, q) = \sum_{n=0}^{\infty} U_n(z). \tag{17}$$

* It will be found that $z \cos mz$ first appears in $U_m(z)$, i.e. in $V_{m+1}(z)$.

8. We now proceed to show that $\sum_2^\infty \lambda_n$ actually vanishes by working out a few particular cases. Suppose we construct the integral which corresponds to $ce_1(z, q)$, the particular value of "A" for which $ce_1(z, q)$ was obtained being given by (11), viz.,

$$A = 1 - 8q - 8q^2 + 8q^3 - \frac{8}{3}q^4 - \frac{88}{9}q^5 + \cdots, \text{ etc.}$$

If we write

$$A = 1 + 8p,$$

Mathieu's differential equation reduces to

$$\frac{d^2y}{dz^2} + y = -8(p + 2q\cos 2z)y.$$

In this case $U_0(z)$ is evidently $\sin z$.

(*a*) To get $U_1(z)$, we express $-8(p + 2q\cos 2z)U_0(z)$ in a series of sines in the form

$$-8(p - q)\sin z - 8q\sin 3z = V_1(z),$$

and solve the equation

$$\frac{d^2y}{dz^2} + y = W_1(z),$$

where $W_1(z) = V_1(z) + 8(p + q)\sin z$, since there is no term $z\cos z$ contained in $V_1(z)$.

The integral of the above equation is found to be

$$U_1(z) = -8qz\cos z + q\sin 3z.$$

(*b*) Again, we express $-8(p + 2q\cos z)U_1(z)$ in a series of sines of multiples of z, thus

$$64q(p + q)z\cos z + 64q^2z\cos 3z - 8pq\sin 3z - 8q^2\sin 5z - 8q^2\sin z$$

which we denote by $V_2(z)$.

Then we shall have to find an integral of the differential equation

$$\frac{d^2y}{dz^2} + y = W_2(z),$$

where $W_2(z) = V_2(z) - 64q(p + q)z\cos z + 8q^2\sin z$ (here $W_2(z)$ is made independent of $z\cos z$). On simplification, we get

$$W_2(z) = 64q^2z\cos 3z - 8pq\sin 3z + 8q^2\sin 5z.$$

The integral of the above equation is given by

$$U_2(z) = -8q^2z\cos 3z + q(6q + p)\sin 3z + \frac{1}{3}q^2\sin 5z.$$

(*c*) Again, since

$$- 8(p + 2q \cos 2z)U_2(z)$$
$$= 64q^3z \cos 5z + 64pq^2z \cos 3z + 64q^3z \cos z$$
$$- 8q^2(6q + p) \sin z - \{8pq(6q + p) + \frac{8}{3}q^3\} \sin 3z$$
$$- 8\{q^2(6q + p) + \frac{1}{3}pq^2\} \sin 5z - \frac{8}{3}q^3 \sin 7z,$$

which we denote by $V_3(z)$, we have

$$W_3(z) = V_3(z) - 64q^3z \cos z - 8q^3 \sin z,$$

so that $W_3(z)$ may not contain $z \cos z$.

Now, solving the differential equation

$$\frac{d^2y}{dz^2} + y = W_3(z),$$

we find

$$U_3(z) = - \frac{8}{3}q^3z \cos 5z - 8pq^2z \cos 3z + 4q^2(7q + p)z \cos z$$
$$+ q\left(p^2 + 12pq + \frac{1}{3}q^2\right) \sin 3z + \frac{4}{9}q^2(p + 7q) \sin 5z + \frac{1}{18}q^3 \sin 7z.$$

(*d*) In the same way, $V_4(z) = - 8(p + 2q \cos 2z)U_3(z)$. Now, if we express it in a series of sines, we find that the term which contains $z \cos z$ as a factor is

$$- 32q^2(p^2 + 6pq + 7q^2)z \cos z,$$

and the term which contains $\sin z$ is

$$- 8q^2\left(p^2 + 12pq + \frac{1}{3}q^2\right) \sin z.$$

Hence we define $W_4(z)$ such that

$$W_4(z) = V_4(z) + 32q^2(p + 6pq + 7q^2)z \cos z + \frac{8}{3}q^4 \sin z.$$

$W_4(z)$ is thus made independent of $z \cos z$. We can now find the integral $U_4(z)$.

Proceeding thus, we can find all the integrals $U_0(z)$, $U_1(z)$, $U_2(z)$, $U_3(z)$, $\cdots U_n(z)$, $\cdots$, etc. Hence if, in the series

$$U_0(z) + U_1(z) + U_2(z) + \cdots + U_n(z) + \cdots,$$

as found above, we substitute by (11)

$$8p = - 8q - 8q^2 + 8q^3 - \frac{8}{3}q^4 - \cdots, \text{ etc.},$$

we get, after arranging,

$$- 8q(1 - 3q^2 + \cdots)z\{\cos z + q \cos 3z + q^2(- \cos 3z + \tfrac{1}{3} \cos 5z) + \cdots\}$$
$$+ \sin z + q \sin 3z + q^2(\tfrac{1}{3} \sin 5z + 5 \sin 3z)$$
$$+ q^3\left(\frac{1}{18} \sin 7z + \frac{8}{3} \sin 5z - \frac{35}{3} \sin 3z\right) + \cdots \tag{18}$$

correct to the third power of q.

And here it will be noticed that $\sum_2^\infty \lambda_n$ as obtained above is given by

$$- 64q(p + q) - 64q^3 + 32q^2(p^2 + 6pq + 7q^2) + \cdots,$$

which, on substitution of the value of "p" in terms of "q," vanishes if we neglect terms containing higher powers of "q" than the fourth.

Hence the series (18) is a solution of the differential equation (1) and is the solution of the second kind denoted by $i\eta_1(z, q)$.

9. Similarly if we proceed to construct the integral corresponding to the value of "A" for $ce_2(z, q)$, viz.,

$$A = 4 + \frac{80}{3} q^2 - \frac{6104}{27} q^4 + \cdots, \text{ etc.,}$$

we find

$$U_0(z) = \sin 2z,$$
$$U_1(z) = \frac{2}{3} q \sin 4z,$$
$$U_2(z) = \frac{1}{6} q^2 \sin 6z + \frac{4}{9} pq \sin 4z + 8q^2 z \cos 2z,$$
$$U_3(z) = \frac{1}{45} q^3 \sin 8z + \frac{11}{72} pq^2 \sin 6z + \frac{1}{9} q\left(\frac{8}{3} p^2 - 31q^2\right) \sin 4z$$
$$- 16q^3 z + \tfrac{8}{9} pq^2 z \cos 2z + \tfrac{16}{3} q^3 z \cos 4z,$$

. . . .

. . . .

and

$$\sum \lambda_n = 64pq^2 + \left(\frac{64}{9} p^2 q^2 - 256q^4 + \frac{128}{3} q^4\right) + \cdots, \text{ etc.}$$
$$= 8q^2\left(\frac{80}{3} q^2 + \cdots\right) + \frac{1}{9} q^2 \left(\frac{80}{3} q^3 + \cdots\right)^2 - 256q^4 + \frac{128}{3} q^4 + \cdots$$
$$= 0, \text{ neglecting higher powers of } q \text{ than the fourth.}$$

Thus $\sum \lambda_n$ vanishes and $\sum U_n(z)$ gives the solution of the second kind corresponding to $ce_2(z, q)$. It is evidently the solution $i\eta_2(z, q)$.

10. Again, if we proceed to construct the integral of the second kind corresponding to the integral $ce_3(z, q)$, for which

$$A = 9 + 4q^2 - 8q^3 + \frac{13}{5} q^4 + \cdots, \text{ etc.,}$$

we obtain, in this case,

$$U_0(z) = \sin 3z,$$

$$U_1(z) = \frac{1}{2} q \sin 5z - q \sin z,$$

$$U_2(z) = \frac{1}{10} q^2 \sin 7z + \frac{1}{4} pq \sin 5z + q(p - q) \sin z,$$

$$U_3(z) = \frac{1}{90} q^3 \sin 9z + \frac{7}{100} pq^2 \sin 7z + \frac{1}{8} q \left(p^2 + \frac{2}{5} q^2 \right) \sin 5z + \frac{1}{3} q^2 (5p - 8q) z \cos 3z - q(p - q)^2 \sin z,$$

. . . .

. . . .

and

$$\sum \lambda_n = \frac{8}{3} pq^2 (5p - 8q) + \left(\frac{32}{5} q^5 - \frac{40}{3} pq^4 + \frac{64}{3} p^2 q^3 - \frac{28}{3} p^3 q^2 \right) + \cdots, \text{ etc.}$$

$$= \frac{5}{24} \{4q^2 - 8q^3 + \cdots\}^2 q^2 - \frac{8}{3} \{4q^2 - 8q^3 + \cdots\} q^3 + \frac{32}{5} q^5 - \frac{5}{3} \{4q^2 - 8q^3 + \cdots\} q^4 + \frac{1}{3} \{4q^2 - 8q^3 + \cdots\}^2 q^3 - \frac{7}{384} \{4q^2 - 8q^3 + \cdots\}^3 q^2 + \cdots, \text{ etc.}$$

$= 0$, neglecting higher powers of q than the fifth.

Hence $\sum U_n(z)$ as obtained from above is the second solution corresponding to $ce_3(z, q)$ and is denoted by $i\eta_3(z, q)$.

That $\sum \lambda_n$ vanishes, has been further verified in a few other cases.

ON THE CONVERGENCE OF THE SERIES OF INTEGRALS OF THE SECOND KIND.

11. The process of term-by-term differentiation which we have carried out in § 7 is only permissible when we have proved that the infinite series $\sum U_n(z)$ is a uniformly convergent series of analytic functions. It is, therefore, necessary for us to examine the solution $\sum U_n(z)$ more closely with a view to study its convergence.

The forms of $U_n(z)$ which are solutions of the differential equation

$$\frac{d^2 y}{dz^2} + m^2 y = W_n(z)$$

will be of the following types:

(i) when $n < m$,

$$U_n(z) = {}^* \sum_{r=1}^{n}{}' \beta_{n,\,r} \sin (m - 2r) z + \sum_{r=1}^{m} \alpha_{n,\,r} \sin (m + 2r) z,$$

* Σ' means that the summation ceases at the greatest value of r, which is less than or equal to $m/2$.

(ii) when $n = m$,

$$U_m(z) = *\sum_{r=1}^{m}{}' \beta_{m,\,r} \sin (m-2r)z + \sum_{r=1}^{m} \alpha_{m,\,r} \sin (m+2r)z + \delta_{0,\,0} z \cos mz,$$

(iii) when $n > m$, i.e., when $n = m + \eta$ and $\eta \geq 1$,

$$U_{m+\eta}(z) = *\sum_{r=1}^{m+\eta}{}' \beta_{m+\eta,\,r} \sin (m-2r)z + \sum_{r=1}^{m+\eta} \alpha_{m+\eta,\,r} \sin (m+2r)z$$

$$+ z\left\{\sum_{r=1}^{\eta}{}' \gamma_{\eta,\,r} \cos (m-2r)z + \sum_{r=0}^{\eta} \delta_{\eta,\,r} \cos (m+2r)z\right\}.$$

Then since

$$\left(\frac{d^2}{dz^2} + m^2\right) \cdot U_{m+\eta+1}(z) = -8(p + 2q\cos 2z)U_{m+\eta}(z) - \lambda_{m+\eta+1} z \cos mz$$

$$- a_n q^n \sin mz,$$

we get on equating the coefficients of $z \cos (m + 2r)z$, $z \cos (m - 2r)z$, $\sin (m + 2r)z$, and $\sin (m - 2r)z$, the following recurrence-formulæ:

$$(a) \quad r(m+r)\delta_{\eta+1,\,r} = 2\{p\delta_{\eta,\,r} + q(\delta_{\eta,\,r-1} + \delta_{\eta,\,r+1})\}, \quad (r = 1, 2, 3, \cdots)$$

$$(b) \quad r(r-m)\gamma_{\eta+1,\,r} = 2\{p\gamma_{\eta,\,r} + q(\gamma_{\eta,\,r-1} + \gamma_{\eta,\,r+1})\}, \quad \left(r \leq \frac{m}{2}\right)$$

$$(c) \quad r(m+r)\alpha_{m+\eta+1,\,r} + \frac{1}{2}(m+2r)\delta_{\eta+1,\,r}$$

$$= 2\{p\alpha_{m+\eta,\,r} + q(\alpha_{m+\eta,\,r+1} + \alpha_{m+\eta,\,r-1})\} \quad \left(\begin{matrix} r = 1, 2, 3, \cdots \\ \eta \geq 1 \end{matrix}\right),$$

but when $n < m$,

$$r(m+r)\alpha_{n+1,\,r} = 2\{p\alpha_{n,\,r} + q(\alpha_{n,\,r+1} + \alpha_{n,\,r-1})\},$$

$$(d) \quad r(r-m)\beta_{m+\eta+1,\,r} + \frac{1}{2}(m-2r)\gamma_{\eta+1,\,r}$$

$$= 2\{p\beta_{m+\eta,\,r} + q(\beta_{m+\eta,\,r+1} + \beta_{m+\eta,\,r-1})\}, \quad \left(r \leq \frac{m}{2}\right),$$

but when $n < m$,

$$r(r-m)\beta_{n+1,\,r} = 2\{p\beta_{n,\,r} + q(\beta_{n,\,r-1} + \beta_{n,\,r+1})\}, \quad \left(r \leq \frac{m}{2}\right),$$

with the following restrictions:

$$\alpha_{n,\,r} = \beta_{n,\,r} = \gamma_{n,\,r} = \delta_{n,\,r} = 0, \quad \text{if} \quad r > n,$$

and also

$$\alpha_{n,\,0} = \beta_{n,\,0} = \gamma_{n,\,0} = 0, \quad \text{whatever } n \text{ is.}$$

12. If we denote

$$\sum_{\eta=r}^{\infty} \delta_{\eta,\,r} = D_r, \qquad \sum_{\eta=r}^{\infty} \gamma_{\eta,\,r} = C_r,$$

then D_r and C_r give the sum of the coefficients of $z \cos (m + 2r)z$, and $z \cos (m - 2r)z$ respectively in $\sum U_n(z)$.

Hence from (*a*) and (*b*), § 11, we get

$$r(m + r)D_r = 2\{pD_r + q(D_{r-1} + D_{r+1})\}, \tag{18}$$
$$r(r - m)C_r = 2\{pC_r + q(C_{r-1} + C_{r+1})\}. \tag{19}$$

From the forms given in (18) and (19), it is evident that the series for D_r and C_r are both convergent* and that

$$\lim_{r=\infty} D_r = 0, \qquad \text{and} \qquad \lim_{r=\infty} C_r = 0.$$

Similarly, denoting

$$\sum_{n=1}^{\infty} \alpha_{n,\,r} = A_r; \qquad \sum_{n=r}^{\infty} \beta_{n,\,r} = B_r,$$

which represent respectively the coefficients of sin $(m + 2r)z$ and sin $(m - 2r)z$, occurring in $\sum U_n(z)$, we get from (*c*) and (*d*)

$$r(m + r)A_r + \frac{1}{2}(m + 2r)D_r = 2\{pA_r + q(A_{r-1} + A_{r+1})\}, \tag{20}$$

$$r(r - m)B_r + \frac{1}{2}(m - 2r)C_r = 2\{pB_r + q(B_{r-1} + B_{r+1})\}. \tag{21}$$

Writing

$$\omega_r = -2q/\{r(m + r) - 2p\}, \qquad \omega'_r = -2q/\{r(r - m) - 2p\},$$
$$v_r = (m + 2r)/2\{r(m + r) - 2p\}, \qquad v'_r = (m - 2r)/2\{r(m + r) - 2p\},$$

we get from (20) and (21) respectively

$$v_r D_r + \omega_r A_{r-1} + A_r + \omega_r A_{r+1} = 0, \tag{22}$$
$$v'_r C_r + \omega'_r B_{r-1} + B_r + \omega'_r B_{r+1} = 0. \tag{23}$$

Eliminating $A_1, A_2, \cdots A_{r-1}, A_{r+1}, \cdots$ from (22), we get

$$A_r = \frac{(-1)^r}{\triangle_0} \square_r, \tag{24}$$

where

$$\triangle_0 = \begin{vmatrix} 1 & \omega_1 & & & & & & & \\ \omega_2 & 1 & \omega_2 & & & & & & \\ 0 & \omega_3 & 1 & \omega_3 & & & & & \\ 0 & 0 & \omega_4 & 1 & \omega_4 & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

* Whittaker and Watson, "Modern Analysis," pp. 415–416.

$$\square_r = \begin{vmatrix} v_1D_1+\omega_1, & 1, & \omega_1, & & & & & & & & \\ v_2D_2, & \omega_2, & 1, & \omega_2, & & & & & & & \\ v_3D_3, & 0, & \omega_3, & 1, & \omega_3, & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ v_{r-1}D_{r-1}, & 0 & 0 & 0 & 0 & \omega_{r-1}, & 1, & 0 & & & \\ v_rD_r, & 0 & 0 & 0 & 0 & 0 & \omega_r, & \omega_r, & 0 & & \\ v_{r+1}D_{r+1}, & 0 & 0 & 0 & 0 & 0 & 0 & 1, & \omega_{r+1}, & 0 & \\ v_{r+2}D_{r+2}, & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{r+2}, & 1, & \omega_{r+2}, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

since $A_0 = 1$.

Expanding the determinant $\square_r$ in terms of the elements of the first column, we get

$$A_r = \frac{(-1)^r}{\triangle_0}\left\{(v_1D_1+\omega_1)M_1 + \sum_{k=2}^{\infty} v_kD_kM_k\right\}, \tag{25}$$

where the M's denote the first minors of the elements of the first column.

13. It will be easily seen that with $s \geq 1$

(i) $$M_{r+s} = (-1)^{r+s-1}\cdot\omega_r\omega_{r+1}\cdots\omega_{r+s-1}\cdot\triangle_{r+s}\begin{vmatrix} 1 & \omega_1 & & & & \\ \omega_2 & 1 & \omega_2 & & & \\ 0 & \omega_3 & 1 & \omega_3 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \omega_{r-1} & 1 \end{vmatrix},$$

where $\triangle_{r+s}$ stands for the infinite determinant

$$\begin{vmatrix} 1 & \omega_{r+s+1} & & & & & \\ \omega_{r+s+2} & 1 & \omega_{r+s+2} & & & & \\ 0 & \omega_{r+s+3} & 1 & \omega_{r+s+3} & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

and also when $k < r$

(ii) $$M_k = (-1)^{k-1}\omega_{k+1}\omega_{k+2}\cdots\omega_r\cdot\triangle_r\begin{vmatrix} 1 & \omega_1 & & & & \\ \omega_2 & 1 & \omega_2 & & & \\ 0 & \omega_3 & 1 & \omega_3 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \omega_{k-1} & 1 \end{vmatrix},$$

and further

$$\text{(iii)}\quad M_r = (-1)^{r-1}\begin{vmatrix} 1 & \omega_1 & & & & & & & & \\ \omega_2 & 1 & \omega_2 & & & & & & & \\ 0 & \omega_3 & 1 & \omega_3 & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \omega_{r-1} & 1 & 0 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \omega_{r+1} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega_{r+2} & 1 & \omega_{r+2} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The infinite determinant $\triangle_r$ is convergent, whatever 'r' is, and further that $\lim\limits_{r=\infty} \triangle_r = 1$. Hence the M's are all finite and the series (25) is therefore convergent and converges to a finite value, if r is finite, but vanishes if r is made infinitely large.

The same may be proved for B_r by means of the relation (23).

The series $\sum U_n$ is, therefore, uniformly convergent in any bounded domain of z so that term-by-term differentiation is permissible.

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* Helge Von Koch, *Acta Mathematica*, Vol. XVI.